# THE EVALUATION OF TORNHEIM DOUBLE SUMS. PART 2

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Abstract. We provide an explicit formula for the Tornheim double series $T(a,0,c)$ in terms of an integral involving the Hurwitz zeta function. For integer values of the parameters, $a = m$, $c = n$, we show that in the most interesting case of *even* weight $N := m+n$ the Tornheim sum $T(m,0,n)$ can be expressed in terms of zeta values and the family of integrals

$$\int_0^1 \log\Gamma(q) B_k(q)\, \mathrm{Cl}_{l+1}(2\pi q)\, dq,$$

with $k+l = N$, where $B_k(q)$ is a Bernoulli polynomial and $\mathrm{Cl}_{l+1}(x)$ is a Clausen function.

## 1. Introduction

The function

$$T(a,b,c) := \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{1}{r^a\, s^b\, (r+s)^c}, \tag{1.1}$$

was introduced by Tornheim in [11]. For $a,\, b,\, c \in \mathbb{R}$, the series is convergent if

$$a+c>1,\; b+c>1, \text{ and } a+b+c>2. \tag{1.2}$$

In the case $(a,b,c) = (m,k,n)$, with $m,\, k,\, n \in \mathbb{N}\cup\{0\}$, we define the *weight* of the Tornheim sum $T(m,k,n)$ as the positive integer $N = m+k+n$.

We have previously derived [7] an analytic expression for the general Tornheim sum $T(a,b,c)$ in terms of integrals involving the Hurwitz zeta function $\zeta(z,q)$, defined as the meromorphic extension to the whole complex $z$-plane of the series

$$\zeta(z,q) := \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}, \tag{1.3}$$

which is defined for $\mathrm{Re}\, z > 1$ and $q \neq 0,\, -1,\, -2,\, \cdots$. Our expression for $T(a,b,c)$, recalled later in Theorem 3.2, is valid for $a,\, b,\, c \in \mathbb{R} - \mathbb{N}\cup\{0\}$, provided the convergence conditions (1.2) are satisfied. Using this result and a limiting procedure, we derived similar formulas for $T(m,k,n)$, with $m,\, k,\, n \in \mathbb{N}$.

In this paper we derive a formula for the Tornheim sum $T(a,0,c)$, valid for $a,\, c \in \mathbb{R}-\mathbb{N}$ with $a>2$ and $c>2$. A limiting procedure will then provide an analytic expression for the sums $T(m,0,n)$, with $m,\, n \in \mathbb{N}-\{1\}$.

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These expressions for the sums $T(m,0,n)$ are of practical interest. In fact, Huard et al [10] established the relation

$$T(m,k,n) = \sum_{i=1}^{m} \binom{m+k-i-1}{m-i} T(i,0,N-i) + \sum_{i=1}^{k} \binom{m+k-i-1}{k-i} T(i,0,N-i), \tag{1.4}$$

for the Tornheim sum of weight $N = m+k+n$. Therefore, it suffices to consider sums of the form $T(m,0,n)$. The convergence of the series requires $n > 1$ and $m+n > 2$, thus the sum $T(m,0,1)$ diverges.

Introduce the spaces

$$\mathcal{Z}_N := \{T(m,k,n) : \, m, \, k, \, n \in \mathbb{N} \cup \{0\} \text{ with } n+m \geq 2, \, k+n \geq 2 \text{ and } N = m+k+n \geq 3\}, \tag{1.5}$$

and

$$\mathcal{Z}_N^0 := \{T(m,0,n) \in \mathcal{Z}_N\}. \tag{1.6}$$

The following result is contained in formula (1.4):

**Proposition 1.1.** Every sum in $\mathcal{Z}_N$ is a linear combination of terms in $\mathcal{Z}_N^0$ with coefficients in $\mathbb{N}$.

**Note 1.2.** The sums $T(m,0,n)$ appearing in (1.4) can be written as

$$\begin{aligned} T(m,0,n) &= \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{r^m \, (r+s)^n} \\ &= \sum_{r_1 > r_2} \frac{1}{r_1^n \, r_2^m}. \end{aligned} \tag{1.7}$$

Therefore $T(m,0,n)$ is a special case of the multiple zeta value (= MZV)

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{r_1 > r_2 > \cdots > r_k > 0} \prod_{j=1}^{k} r_j^{-s_j}, \tag{1.8}$$

namely[1],

$$T(m,0,n) = \zeta(n,m). \tag{1.9}$$

There is a vast literature on MZV and the reader if referred to Chapter 3 of [2] for an introduction to this topic.

**Note 1.3.** In the case of *odd* weight, [10] gives the relation

$$\begin{aligned} T(m,0,n) = (-1)^m & \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{m+n-2j-1}{m-1} \zeta(2j)\zeta(m+n-2j) \\ &+ (-1)^m \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+n-2j-1}{n-1} \zeta(2j)\zeta(m+n-2j) - \tfrac{1}{2}\zeta(m+n), \end{aligned} \tag{1.10}$$

[1] The double zeta function appearing here should not be confused with the Hurwitz zeta function in (1.3).

valid for $m \geq 1$. Here

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.11}$$

is the classical Riemann zeta function. This function has an analytic extension to $\mathbb{C} - \{1\}$, the point $s = 1$ being a simple pole. Recalling that $\zeta(0) = -1/2$, the previous result can also be restated in the following terms:

**Proposition 1.4.** Assume the weight $N$ is odd. Then the sums in $\mathcal{Z}_N^0$ can be evaluated as linear combination of the products $\zeta(2j)\zeta(N-2j)$, $j = \max\left\{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor\right\}$, with integer coefficients.

The idea of [10] is to produce a linear system of equations for the unknowns $T_i := T(i, 0, N-i)$, for $1 \leq i \leq N-2$. This system has full rank in the case $N$ odd and its solution yields (1.10). Their methods also produce analytic expressions for Tornheim sums of small *even* weight, but they fail in general. In particular, for weight $N \geq 8$, the system of equation mentioned above is not of full rank. Section 6 contains a discussion of this issue.

For example, the class $\mathcal{Z}_4^0$ contains the sums

$$T(1,0,3) = \frac{1}{4}\zeta(4) \qquad \text{and} \qquad T(2,0,2) = \frac{3}{4}\zeta(4), \tag{1.12}$$

and in the space $\mathcal{Z}_6^0$ we find the four sums

$$\begin{aligned} T(1,0,5) &= -\frac{1}{2}\zeta(3)^2 + \frac{3}{4}\zeta(6), \\ T(2,0,4) &= \zeta^2(3) - \frac{4}{3}\zeta(6), \\ T(3,0,3) &= \frac{1}{2}\zeta(3)^2 - \frac{1}{2}\zeta(6), \\ T(4,0,2) &= -\zeta^2(3) + \frac{25}{12}\zeta(6). \end{aligned} \tag{1.13}$$

Huard et al. [10] also gave the relation

$$5T(2,0,6) + 2T(3,0,5) = 10\zeta(3)\zeta(5) - \frac{49}{4}\zeta(8), \tag{1.14}$$

for the case of weight 8, but they are unable to evaluate the individual terms $T(2,0,6)$ and $T(3,0,5)$. The question of an analytic expression these sums remains open.

In this paper we give particular consideration to the Tornheim sums $T(m,0,n)$ of arbitrary even weight $N = m+n$. For each even $N$, only two of these sums have known closed expressions in terms of zeta values, namely $T(1,0,N-1)$ and $T(N/2,0,N/2)$.

Tornheim established the result

$$T(0,0,N) = \zeta(N-1) - \zeta(N), \quad N \geq 3, \tag{1.15}$$

which appears as Theorem 5, page 308 of Tornheim [11], and the companion formula

$$T(1,0,N-1) = \frac{1}{2}\left[(N-1)\zeta(N) - \sum_{i=2}^{N-2} \zeta(i)\zeta(N-i)\right], \quad N \geq 3, \tag{1.16}$$

where $N = n+1$, can also be found in [11].

The main result of this paper is an analytic expression for the Tornheim sums $T(m,0,n)$ of *even* weight, with $m$, $n \geq 2$, in terms of a family of integrals involving the *log-gamma* function $\log \Gamma(q)$, the *Bernoulli polynomials* $B_k(q)$, given by the generating function

$$\frac{e^{qt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(q) \frac{t^{k-1}}{k!}, \tag{1.17}$$

and the *Clausen functions* $\mathrm{Cl}_l(x)$, defined as

$$\mathrm{Cl}_{2n}(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n}}, \quad n \in \mathbb{N}, \tag{1.18}$$

$$\mathrm{Cl}_{2n+1}(x) := \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n+1}}, \quad n \in \mathbb{N} \cup \{0\}. \tag{1.19}$$

For example, we obtain

$$T(6,0,2) = \tfrac{7}{6}\zeta(8) - 6\zeta(3)\zeta(5) - Y^*_{2,6}, \tag{1.20}$$

with

$$Y^*_{2,6} := \frac{8}{3}\pi^6 \left( X_{0,6} - 2X_{1,5} + X_{2,4} \right) - 6\zeta(7)\log 2\pi,$$

and where

$$X_{k,l} := (-1)^{\lfloor l/2 \rfloor} \frac{l!}{(2\pi)^l} \int_0^1 \log \Gamma(q) B_k(q) \, \mathrm{Cl}_{l+1}(2\pi q) \, dq. \tag{1.21}$$

In the general case, we show that all the Tornheim sums of even weight $N$ can be expressed in terms of values of the Riemann zeta function and integrals of the form

$$Y^*_{m,N-m} := \frac{2(2\pi)^{N-2}}{m!(N-m-2)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} X_{j,N-2-j} + (-1)^{\frac{N}{2}-1} \binom{N-2}{m-1} \zeta(N-1) \log 2\pi, \tag{1.22}$$

where $N$ is the weight and $m$ is even in the range $2 \leq m \leq 2\left\lfloor \frac{N-2}{6} \right\rfloor$.

The rest of the paper is organized as follows. Section 2 contains all the main theorems we will prove in the subsequent sections. Section 3 derives the expression for the Tornheim sum $T(a,0,c)$ in terms of an integral involving the Hurwitz zeta function, starting from a more general result derived in [7]. Section 4 computes the limit of $T(a,0,c)$ as $a \to m$ and $c \to n$, with $m, n \in \mathbb{N} - \{1\}$. In Section 5 we consider the particular case of Tornheim sums of even weight $N$ and show that they can be expressed in terms of zeta values and the family of integrals $X_{k,l}$, defined above, with $k+l = N$. The explicit evaluation of this last family of definite integrals remains a challenging problem. Finally, in Section 6 we give a systematic list of evaluations for small even weight.

## 2. Main results

We start by introducing some auxiliary special functions that play a role in our derivations. First, we have the *Bernoulli functions*

$$A_k(q) = k\zeta'(1-k,q), \quad k \in \mathbb{N}, \tag{2.1}$$

introduced in [5, 6], and the kernel

$$K(q) = \log \sin \pi q. \tag{2.2}$$

The first result is established in Section 3.

**Theorem 2.1.** Let $a, c \in \mathbb{R} - \mathbb{N}$ with $a,\, c > 2$. Then

$$\begin{aligned} T(a,0,c) = {} & 4\lambda(a)\lambda(c)\sin\left(\frac{\pi c}{2}\right) \\ & \times \left[ \sin\left(\frac{\pi a}{2}\right) \left\{ \zeta(1-a)\zeta(1-c) - \frac{\zeta(1-a-c)B(a,c)}{1-\tan\left(\frac{\pi a}{2}\right)\tan\left(\frac{\pi c}{2}\right)} \right\} \right. \\ & \left. - \frac{1}{2}\cos\left(\frac{\pi a}{2}\right) \int_0^1 \left[\zeta(1-a,q) - \zeta(1-a,1-q)\right] \zeta(1-c,q) \cot \pi q \, dq \right], \end{aligned} \tag{2.3}$$

where

$$\lambda(z) := \frac{\Gamma(1-z)}{(2\pi)^{1-z}} = \frac{\pi}{(2\pi)^{1-z}\,\Gamma(z)\,\sin \pi z}. \tag{2.4}$$

The explicit expression (2.3) for $T(a,0,c)$ allows us to consider the limit $a \to m, c \to n$, for $m, n \in \mathbb{N} - \{1\}$. The value of $T(m,0,n)$ is found to be given in terms of the *basic integrals*:

$$\begin{aligned} I_{BB}(k,l) &:= \int_0^1 B_k(q)B_l(q)K(q)\,dq, \\ I_{AB}(k,l) &:= \frac{1}{\pi}\int_0^1 A_k(q)B_l(q)K(q)\,dq, \\ I_{AA}(k,l) &:= \frac{1}{\pi^2}\int_0^1 A_k(q)A_l(q)K(q)\,dq, \\ J_{AA}(k,l) &:= \frac{1}{\pi^2}\int_0^1 A_k(q)A_l(1-q)K(q)\,dq. \end{aligned} \tag{2.5}$$

where $B_k(q)$ is a Bernoulli polynomial and $A_k(q)$ and $K(q)$ are the functions introduced in (2.1) and (2.2), respectively.

**Theorem 2.2.** Let $m \geq 2$, $n \geq 2 \in \mathbb{N}$. The Tornheim sum $T(m,0,n)$ is given by

$$T(m,0,n) = \zeta(m)\zeta(n) - \tfrac{1}{2}\zeta(m+n) + (-1)^{\lfloor \frac{m+n}{2} \rfloor} \frac{(2\pi)^{m+n-1}}{m!n!} \ell_2(m,n), \tag{2.6}$$

where:

(a) $m$ and $n$ even:

$$\ell_2(m,n) = mI_{AB}(m-1,n) + nI_{AB}(m,n-1), \tag{2.7}$$

(b) $m$ and $n$ odd:

$$\ell_2(m,n) = mI_{AB}(n,m-1) + nI_{AB}(n-1,m). \tag{2.8}$$

(c) $m$ odd and $n$ even:

$$\ell_2(m,n) = \frac{1}{2}\left(mI_{BB}(m-1,n) + nI_{BB}(m,n-1)\right), \tag{2.9}$$

(d) $m$ even and $n$ odd:

$$\begin{aligned}\ell_2(m,n) = &-mI_{AA}(m-1,n) - nI_{AA}(m,n-1)\\ &- mJ_{AA}(m-1,n) + nJ_{AA}(m,n-1),\end{aligned} \tag{2.10}$$

The proof of this theorem is the subject of Section 4.

In the case of even weight $m+n=N$, the Tornheim sum $T(m,0,n)$ is given in terms of the single family of integrals $I_{AB}(k,l)$. The next theorem shows that these can be expressed in terms of the family of integrals

$$X_{k,l} := (-1)^{\lfloor l/2 \rfloor}\frac{l!}{(2\pi)^l}\int_0^1 \log\Gamma(q)B_k(q)\,\mathrm{Cl}_{l+1}(2\pi q)\,dq. \tag{2.11}$$

**Theorem 2.3.** Assume $m,\, n \in \mathbb{N}$ satisfy $m,\, n \geq 2$ and that the weight $N := m+n$ is even. Define

$$T_1(m,n) := \zeta(m)\zeta(n) - \frac{1}{2}\zeta(N) \tag{2.12}$$

and

$$T_2(m,n) := -\sum_{k=1}^{N/2-2}\binom{N-2-2k}{m-1}\zeta(2k+1)\zeta(N-1-2k) + (-1)^{N/2-1}Y^*_{m,n}, \tag{2.13}$$

where

$$\begin{aligned}Y^*_{m,n} := \frac{2(2\pi)^{N-2}}{m!(N-m-2)!}&\sum_{j=0}^{m}(-1)^j\binom{m}{j}X_{j,N-2-j}\\ &+(-1)^{\frac{N}{2}-1}\binom{N-2}{m-1}\zeta(N-1)\log 2\pi\end{aligned} \tag{2.14}$$

$$Y^*_{m,n} := Y_{m,n} + (-1)^{\frac{N}{2}-1}\binom{N-2}{m-1}\zeta(N-1)\log 2\pi.$$

Then $T(m,0,n)$ can be written as

$$T(m,0,n) = \begin{cases} T_1(m,n) + T_2(m,n), & m \text{ and } n \text{ odd},\\ T_1(m,n) + T_2(n,m), & m \text{ and } n \text{ even}.\end{cases} \tag{2.15}$$

It follows that, for even weight $N=m+n$, $T(m,0,n)$ is determined by either $Y_{m,n}$ or $Y_{n,m}$, depending on the parities of $m$ and $n$. Note that both $Y_{m,n}$ and $Y_{n,m}$ are linear combinations of integrals $X_{k,l}$ with fixed $k+l=N-2=m+n-2$.

We discuss this theorem in Section 5.

## 3. An expression for the Tornheim series $T(a,0,c)$

In this section we present an analytic expression for the Tornheim double series $T(a,0,c)$, valid for $a>2$, $c>2$ and $a,\ c\notin\mathbb{N}$. Note that $T(a,0,c)$ will be finite if $a>1$ and $c>1$. We employ here and in Section 4 the shorthand notation

$$\bar{\zeta}(z,q):=\zeta(1-z,q) \quad \text{and} \quad \bar{\zeta}(z):=\zeta(1-z), \quad \text{for } z\neq 0. \tag{3.1}$$

All the properties of the Hurwitz zeta function that we will use can be expressed in terms of the function $\bar{\zeta}(z,q)$ as follows:

$$\frac{d}{dq}\bar{\zeta}(z,q)=(z-1)\bar{\zeta}(z-1,q), \tag{3.2}$$

$$\bar{\zeta}(k,q)=-\frac{1}{k}B_k(q), \tag{3.3}$$

$$\bar{\zeta}'(k,q)=-\frac{1}{k}A_k(q). \tag{3.4}$$

where $B_k(q)$ is a Bernoulli polynomial and $A_k(q)$ is the Bernoulli function (2.1).

The identities derived below appear from integration by parts. The restrictions imposed on the parameters guaranteee that the boundary terms vanish. Recall that the Hurwitz zeta function satisfies the identity

$$\zeta(z,q)=q^{-z}+\zeta(z,1+q),$$

which implies $\zeta(z,0)=\zeta(z,1)$ if $z<0$. Equivalently,

$$\bar{\zeta}(z,0)=\bar{\zeta}(z,1) \quad \text{if} \quad z>1. \tag{3.5}$$

**Theorem 3.1.** Let $a,c\in\mathbb{R}-\mathbb{N}$ with $a,\ c>2$. Then

$$\begin{aligned} T(a,0,c) = {} & 4\lambda(a)\lambda(c)\sin\left(\frac{\pi c}{2}\right) \\ & \times\left[\sin\left(\frac{\pi a}{2}\right)\left\{\bar{\zeta}(a)\bar{\zeta}(c)-\frac{\bar{\zeta}(a+c)B(a,c)}{1-\tan\left(\frac{\pi a}{2}\right)\tan\left(\frac{\pi c}{2}\right)}\right\}\right. \\ & \left. -\frac{1}{2}\cos\left(\frac{\pi a}{2}\right)\int_0^1\left[\bar{\zeta}(a,q)-\bar{\zeta}(a,1-q)\right]\bar{\zeta}(c,q)\cot\pi q\,dq\right]. \end{aligned} \tag{3.6}$$

The proof is obtained by analyzing the behavior as $\varepsilon\to 0$ of $T(a,b,c)$ given in (3.8) below. This was first derived in [7]. The parameter $b$ is changed to $\varepsilon$ in order to remind ourselves that it is small.

**Theorem 3.2.** Let $a,b,c\in\mathbb{R}$, satisfying $a+c>1$, $b+c>1$, and $a+b+c>2$, and define the auxiliary function $\lambda(z)$ as in (2.4),

$$\lambda(z) \;:=\; \frac{\Gamma(1-z)}{(2\pi)^{1-z}}=\frac{\pi}{(2\pi)^{1-z}\,\Gamma(z)\,\sin\pi z}. \tag{3.7}$$

For $a,b,c\notin\mathbb{N}$ we have

$$T(a,b,c)=4\lambda(a)\lambda(b)\lambda(c)\sin\left(\frac{\pi c}{2}\right)Q(a,b,c) \tag{3.8}$$

where

$$\begin{aligned} Q(a,b,c) := {} & \cos\left(\tfrac{\pi}{2}(a-b)\right)\left[J(c,a,b)+J(c,b,a)\right] \\ & -\cos\left(\tfrac{\pi}{2}(a+b)\right)\left[I(a,b,c)+J(a,b,c)\right] \end{aligned} \tag{3.9}$$

and $I(a,b,c)$, $J(a,b,c)$ are the integrals defined by

$$I(a,b,c) := \int_0^1 \zeta(1-a,q)\zeta(1-b,q)\zeta(1-c,q)\,dq \tag{3.10}$$

and

$$J(a,b,c) := \int_0^1 \zeta(1-a,q)\zeta(1-b,q)\zeta(1-c,1-q)\,dq, \tag{3.11}$$

where $\zeta(z,q)$ is the Hurwitz zeta function.

We consider the behavior of each of the factors in (3.8) as $b=\varepsilon \to 0$.

1) The term $\lambda(\varepsilon)$ is regular in view of $\lambda(0)=\frac{1}{2\pi}$. The first few terms of its expansion are

$$\lambda(\varepsilon) = \frac{1}{2\pi} + \frac{(\gamma+\log 2\pi)}{2\pi}\varepsilon + O(\varepsilon^2). \tag{3.12}$$

2) The expansion

$$\cos\left(\frac{\pi}{2}(a\pm\varepsilon)\right) = \cos\frac{\pi a}{2} \mp \frac{\pi}{2}\sin\frac{\pi a}{2}\varepsilon + O(\varepsilon^2) \tag{3.13}$$

is elementary.

3) To obtain the limiting value of $Q(a,\varepsilon,c)$ as $\varepsilon\to 0$, it is required to analyze the behavior of $\bar{\zeta}(\varepsilon,q)$ as it appears in

$$I(a,\varepsilon,c) = \int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(\varepsilon,q)\bar{\zeta}(c,q)\,dq. \tag{3.14}$$

The function $\bar{\zeta}(\varepsilon,q)$ has a pole at $\varepsilon=0$ and its Laurent expansion at that pole is

$$\bar{\zeta}(\varepsilon,q) = -\frac{1}{\varepsilon} - \psi(q) + O(\varepsilon). \tag{3.15}$$

where $\psi(q)=\frac{d}{dq}\log\Gamma(q)$. This expansion is not uniform in $q$, as the difference

$$\bar{\zeta}(\varepsilon,q) - (-1/\varepsilon - \psi(q)) \tag{3.16}$$

blows up as $q\to 0$, for any *fixed* $\varepsilon>0$. Indeed,

$$\psi(q) = -\frac{1}{q} - \gamma + O(q), \tag{3.17}$$

and

$$\bar{\zeta}(\varepsilon,q) = q^{\varepsilon-1} + \zeta(1-\varepsilon) + O(q), \tag{3.18}$$

showing that (3.16) is not bounded as $q\to 0$. This issue is resolved by shifting the second argument of the integral in (3.14). Lemma 3.3 gives the integral $I(a,\varepsilon,c)$ in terms of integrals of the form $I(a',1+\varepsilon,c')$. The integrand here contains the factor $\bar{\zeta}(1+\varepsilon,q)=\zeta(-\varepsilon,q)$, whose expansion involves $\log\Gamma(q)$, producing a similar blow up. Lemma 3.4 shifts the second argument again, and now $I(a,\varepsilon,c)$ is expressed in terms of $I(a',2+\varepsilon,c')$. The integrand now contains $\bar{\zeta}(2+\varepsilon,q)=\zeta(-1-\varepsilon,q)$ and a uniform expansion is finally achieved.

**Lemma 3.3.** For $\varepsilon > 0$ and $a,\, c > 1$ we have

$$I(a,\varepsilon,c) = -\frac{1}{\varepsilon}\left[(a-1)I(a-1,\varepsilon+1,c) + (c-1)I(a,\varepsilon+1,c-1)\right]. \tag{3.19}$$

Similarly,

$$J(a,\varepsilon,c) = -\frac{1}{\varepsilon}\left[(a-1)J(a-1,\varepsilon+1,c) - (c-1)J(a,\varepsilon+1,c-1)\right]. \tag{3.20}$$

*Proof.* The identity

$$\bar{\zeta}(\varepsilon,q) = \frac{1}{\varepsilon}\frac{d}{dq}\bar{\zeta}(1+\varepsilon,q) \tag{3.21}$$

yields, for $\varepsilon > 0$,

$$\begin{aligned} I(a,\varepsilon,c) &= \frac{1}{\varepsilon}\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c,q)\frac{d}{dq}\bar{\zeta}(1+\varepsilon,q)\,dq \\ &= \frac{1}{\varepsilon}\bar{\zeta}(a,q)\bar{\zeta}(c,q)\bar{\zeta}(1+\varepsilon,q)\Big|_0^1 - \frac{1}{\varepsilon}\int_0^1 \bar{\zeta}(1+\varepsilon,q)\frac{d}{dq}\left[\bar{\zeta}(a,q)\bar{\zeta}(c,q)\right]\,dq. \end{aligned}$$

Now observe that the boundary terms vanish for $\varepsilon > 0$ and $a,\, c > 1$, according to the identity (3.5). The identity for $J$ follows along the same lines. □

The expansion of $I(a,\varepsilon,c)$ requires to iterate the result of Lemma 3.3 twice.

**Lemma 3.4.** For $\varepsilon > 0$ and $a,\, c > 2$,

$$\begin{aligned} I(a,\varepsilon,c) = \frac{1}{\varepsilon(\varepsilon+1)}\big[&(a-1)(a-2)I(a-2,\varepsilon+2,c) \\ &+ 2(a-1)(c-1)I(a-1,\varepsilon+2,c-1) + (c-1)(c-2)I(a,\varepsilon+2,c-2)\big], \end{aligned}$$

with similar expressions for $J(a,\varepsilon,c)$ and $J(c,a,\varepsilon)$.

Replacing in the expression (3.8) gives

$$T(a,\varepsilon,c) = \frac{4\lambda(a)\lambda(\varepsilon)\lambda(c)}{\varepsilon(\varepsilon+1)}\sin\left(\frac{\pi c}{2}\right)LT(a,\varepsilon,c) \tag{3.22}$$

where

$$LT(a,\varepsilon,c) = \cos\left(\frac{\pi}{2}(a-\varepsilon)\right)H_+(a,\varepsilon,c) - \cos\left(\frac{\pi}{2}(a+\varepsilon)\right)H_-(a,\varepsilon,c), \tag{3.23}$$

with

(3.24)

$$\begin{aligned} H_+(a,\varepsilon,c) &= (1-a)(2-a)J(c,a-2,\varepsilon+2) + 2(1-a)(1-c)J(c-1,a-1,\varepsilon+2) \\ &+ (1-c)(2-c)J(c-2,a,\varepsilon+2) + (1-c)(2-c)J(c-2,\varepsilon+2,a) \\ &- 2(1-a)(1-c)J(c-1,\varepsilon+2,a-1) + (1-a)(2-a)J(c,\varepsilon+2,a-2), \end{aligned}$$

and

(3.25)

$$\begin{aligned} H_-(a,\varepsilon,c) &= (1-a)(2-a)I(a-2,\varepsilon+2,c) + 2(1-a)(1-c)I(a-1,\varepsilon+2,c-1) \\ &+ (1-c)(2-c)I(a,\varepsilon+2,c-2) + (1-a)(2-a)J(a-2,\varepsilon+2,c) \\ &- 2(1-a)(1-c)J(a-1,\varepsilon+2,c-1) + (1-c)(2-c)J(a,\varepsilon+2,c-2). \end{aligned}$$

The last factor in (3.22) is now expanded in powers of $\varepsilon$ to obtain

$$LT(a,\varepsilon,c) = C_a(M_+ - M_-) + \varepsilon\left[C_a(N_+ - N_-) + \frac{\pi}{2}S_a(M_+ + M_-)\right] + O(\varepsilon^2), \tag{3.26}$$

with a self-explanatory notation. For example $M_+$ is the limit as $\varepsilon \to 0$ of the expression multiplying $\cos(\frac{\pi}{2}(a-\varepsilon))$ in (3.23), $N_+$ is the term of order $\varepsilon$ in the same factor and $C_a = \cos(\pi a/2)$. We now reduce all the terms in (3.26).

**Proposition 3.5.** The identity $M_+ = M_-$ holds. Therefore, the term $LT(a,\varepsilon)$ is of order $\varepsilon$, and we conclude that $T(a,\varepsilon,c)$, in (3.22), is non-singular as $\varepsilon \to 0$.

*Proof.* The proof of this result begins with

(3.27)

$$\begin{aligned} M_+ &= (1-a)(2-a)J(c,a-2,2) + 2(1-a)(1-c)J(c-1,a-1,2) \\ &+ (1-c)(2-c)J(c-2,a,2) + +(1-c)(2-c)J(c-2,2,a) \\ &- 2(1-a)(1-c)J(c-1,2,a-1) + (1-a)(2-a)J(c,2,a-2), \end{aligned}$$

and

$$\begin{aligned} M_- &= (1-a)(2-a)I(a-2,2,c) + 2(1-a)(1-c)I(a-1,2,c-1) \\ &+ (1-c)(2-c)I(a,2,c-2) + (1-a)(2-a)J(a-2,2,c) \\ &- 2(1-a)(1-c)J(a-1,2,c-1) + (1-c)(2-c)J(a,2,c-2). \end{aligned}$$

We now establish some relations for the integrals $I$ and $J$. These are then employed to show that $M_+ = M_-$.

**Lemma 3.6.** The integrals $I$ and $J$ satisfy

$$J(z,z',2) = I(z,z',2) \text{ and } J(z,2,z') = J(z',2,z). \tag{3.28}$$

*Proof.* Start with

$$J(z,z',2) = \int_0^1 \bar{\zeta}(z,q)\bar{\zeta}(z',q)\bar{\zeta}(2,1-q)\,dq, \tag{3.29}$$

with $\bar{\zeta}(2,q) = \zeta(-1,q)$. From $\zeta(1-k,q) = -B_k(q)/k$ and the symmetry of $B_2$ we obtain

$$\bar{\zeta}(2,1-q) = \zeta(-1,1-q) = \zeta(-1,q) = \bar{\zeta}(2,q). \tag{3.30}$$

The first identity follows from there. A similar argument establishes the second one. □

A direct calculation using Lemma 3.6, now shows that $M_+ = M_-$ in (3.26). □

**Calculation of** $M_+$. The next step is to simplify the expression for $M_+$ given in (3.27).

**Proposition 3.7.** Let $a > 2$, $c > 2 \in \mathbb{R}$. Then

$$M_+ = 2\left(\bar{\zeta}(a)\bar{\zeta}(c) - \frac{\bar{\zeta}(a+c)B(a,c)}{1-\tan(\pi a/2)\tan(\pi c/2)}\right). \tag{3.31}$$

*Proof.* Start by producing an alternative form of the term $J(c, a-2, 2)$ given by

$$\begin{aligned}
J(c, a-2, 2) &= \int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(a-2,q)\bar{\zeta}(2,1-q)\,dq \\
&= \frac{1}{a-2}\int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(2,1-q)\frac{d}{dq}\bar{\zeta}(a-1,q)\,dq \\
&= \frac{1}{2-a}\int_0^1 \bar{\zeta}(a-1,q)\frac{d}{dq}\left[\bar{\zeta}(c,q)\bar{\zeta}(2,1-q)\right]\,dq \\
&= \frac{1}{2-a}\int_0^1 \bar{\zeta}(a-1,q)\left[(c-1)\bar{\zeta}(c-1,q)\bar{\zeta}(2,1-q)\right. \\
&\qquad\qquad\left. +\bar{\zeta}(c,q)\bar{\zeta}(1,1-q)\right]\,dq.
\end{aligned}$$

Now recall that

$$\bar{\zeta}(1,q) = -\bar{\zeta}(1,1-q),$$

since $\bar{\zeta}(1,q) = -B_1(q) = \frac{1}{2} - q$, and also

$$\bar{\zeta}(2,q) = \bar{\zeta}(2,1-q),$$

as shown before. Conclude that

$$(a-2)J(c,a-2) = (1-c)I(a-1,c-1,2) + I(a-1,c,1). \tag{3.32}$$

Reversing the order of $a$ and $c$ and considering the expression in (3.27), the first three terms of $M_+$ in (3.27) are reduced to

$$(1-a)I(a-1,c,1) + (1-c)I(c-1,a,1). \tag{3.33}$$

A similar analysis gives an expression for the last three terms in (3.27). This yields a formula for $M_+$ in terms of the integrals $I$ and $J$ where one of the parameters is 1.

**Lemma 3.8.** The term $M_+$ in (3.27) is given by

$$M_+ = (1-a)I(a-1,c,1)+(1-c)I(c-1,a,1)+(1-a)J(a-1,1,c)-(1-c)J(a,1,c-1).$$

We now consider the integrals appearing in this expression for $M_+$.

**Lemma 3.9.** The integral $J$ satisfies

$$J(u,1,v) = -J(v,1,u), \tag{3.34}$$

*Proof.* This comes directly from $\bar{\zeta}(1,1-q) = -\bar{\zeta}(1,q)$. □

The expression for $M_+$ in Lemma 3.8 is now simplified.

**Lemma 3.10.** Let $a > 2$, $c > 2 \in \mathbb{R}$. Then

$$(1-a)I(a-1,c,1) + (1-c)I(a,c-1,1) = \bar{\zeta}(a)\bar{\zeta}(c) - \int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c,q)\,dq$$

and

$$(1-a)J(a-1,1,c) - (1-c)J(a,1,c-1) = \bar{\zeta}(a)\bar{\zeta}(c) - \int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c,1-q)\,dq.$$

Therefore

$$M_+ = 2\bar{\zeta}(a)\bar{\zeta}(c) - \int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c,q)\,dq - \int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c,1-q)\,dq. \tag{3.35}$$

*Proof.* Start with

$$(1-a)I(a-1,c,1) = (1-a)\int_0^1 \bar{\zeta}(a-1,q)\bar{\zeta}(c,q)\bar{\zeta}(1,q)\,dq$$

and the identity $(a-1)\bar{\zeta}(a-1,q) = \frac{d}{dq}\bar{\zeta}(a,q)$. Integration by parts and the value $\bar{\zeta}(1,q) = \frac{1}{2} - q$ give the first identity. The rest of the formulas are established in a similar form. □

The identites

$$\int_0^1 \zeta(1-a,q)\zeta(1-c,q)\,dq = \zeta(1-a-c)B(a,c)\frac{\cos(\frac{\pi}{2}(a-c))}{\cos(\frac{\pi}{2}(a+c))},$$

and

$$\int_0^1 \zeta(1-a,q)\zeta(1-c,1-q)\,dq = \zeta(1-a-c)B(a,c)$$

appear in Theorem 3.1 in [4]. Replacing in (3.35) produces the final expression for $M_+$ claimed in (3.31). □

**Calculation of $N_+$.** This is defined in (3.26) as the term of order $\varepsilon$ in the expansion of $H_+(a,\varepsilon,c)$ defined in (3.24). We illustrate the general method of calculation by computing the term of order $\varepsilon$ in the integral:

$$J(c,a-2,\varepsilon+2) = \int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(a-2,q)\zeta(-1-\varepsilon,1-q)\,dq.$$

The term of order $\varepsilon$ is

$$-\int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(a-2,q)\zeta'(-1,1-q)\,dq$$
$$= -\frac{1}{a-2}\int_0^1 \left(\frac{d}{dq}\bar{\zeta}(a-1,q)\right)\bar{\zeta}(c,q)\zeta'(-1,1-q)\,dq.$$

Integrate by parts and observe that the boundary terms vanish to see that the term of order $\varepsilon$ is

$$\frac{1}{a-2}\int_0^1 \bar{\zeta}(a-1,q)\frac{d}{dq}\left[\bar{\zeta}(c,q)\zeta'(-1,1-q)\right]\,dq.$$

Thus,

$$(a-2)\times \text{ the term of order } \varepsilon \text{ in } J(a-2,c,\varepsilon+2)$$

is

$$(c-1)\int_0^1 \bar{\zeta}(c-1,q)\bar{\zeta}(a-1,q)\zeta'(-1,1-q)\,dq$$
$$-\int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(a-1,q)\frac{d}{du}\zeta'(-1,u)\Big|_{u=1-q}\,dq.$$

Now use

$$\frac{d}{du}\zeta'(-1,u) = \frac{\partial}{\partial z}\Big|_{z=-1}\frac{\partial}{\partial u}\zeta(z,u) = \frac{\partial}{\partial z}\Big|_{z=-1}(-z\zeta(z+1,u)) = -\zeta(0,u)+\zeta'(0,u),$$

to check that the contribution coming from $\zeta'(-1,1-q)$ vanishes and to verify the result described next.

**Lemma 3.11.** The part of $N_+$ coming from integrals with $\varepsilon+2$ in the third variable is

$$-(1-a)\left[\int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(a-1,q)\zeta(0,1-q)\,dq - \int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(a-1,q)\zeta'(0,1-q)\,dq\right]$$
$$-(1-c)\left[\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c-1,q)\zeta(0,1-q)\,dq - \int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c-1,q)\zeta'(0,1-q)\,dq\right].$$

Similar calculations produce the other parts of $N_+$. We spare the reader the details.

**Proposition 3.12.** The term $N_+$ is given by

$$\begin{aligned}2\bar{\zeta}(a)\bar{\zeta}(c) &- \int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c,q)\,dq - \int_0^1 \bar{\zeta}(a,1-q)\bar{\zeta}(c,q)\,dq \\ &+ (1-a)\left[\int_0^1 \bar{\zeta}(c,q)\bar{\zeta}(a-1,q)\zeta'(0,1-q)\,dq \right.\\ &\qquad\left. + \int_0^1 \bar{\zeta}(c,1-q)\bar{\zeta}(a-1,q)\zeta'(0,1-q)\,dq\right] \\ &+ (1-c)\left[\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c-1,q)\zeta'(0,1-q)\,dq \right.\\ &\qquad\left. - \int_0^1 \bar{\zeta}(a,1-q)\bar{\zeta}(c-1,q)\zeta'(0,q)\,dq\right].\end{aligned}$$

The process is now repeated to produce a similar expression for $N_-$. The result is stated next.

**Proposition 3.13.** The difference $N_+ - N_-$ is given by

$$\begin{aligned}N_+ - N_- = (1-a)&\int_0^1 \bar{\zeta}(a-1,q)\bar{\zeta}(c,q)\left[\zeta'(0,q)+\zeta'(0,1-q)\right]\,dq \\ + (1-a)&\int_0^1 \bar{\zeta}(a-1,q)\bar{\zeta}(c,1-q)\left[\zeta'(0,q)+\zeta'(0,1-q)\right]\,dq \\ + (1-c)&\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c-1,q)\left[\zeta'(0,q)+\zeta'(0,1-q)\right]\,dq \\ - (1-c)&\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c-1,1-q)\left[\zeta'(0,q)+\zeta'(0,1-q)\right]\,dq.\end{aligned}$$

The difference $N_+ - N_-$ is now given in the form stated in Theorem 2.1. Recall that that $\zeta'(0,q) = \log\Gamma(q) - \log\sqrt{2\pi}$, so that

$$\zeta'(0,q)+\zeta'(0,1-q) = \log\left[\Gamma(q)\Gamma(1-q)\right] - \log 2\pi = -\log(2\sin\pi q), \tag{3.36}$$

where we have used the reflection rule for the Gamma function

$$\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin\pi q}. \tag{3.37}$$

Integration by parts gives

$$(1-a)\int_0^1 \bar{\zeta}(a-1,q)\bar{\zeta}(c,q)\,dq = -(1-c)\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c-1,q)\,dq,$$

and

$$(1-a)\int_0^1 \bar{\zeta}(a-1,q)\bar{\zeta}(c,1-q)\,dq = (1-c)\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(c-1,1-q)\,dq.$$

These identities show that in the expansion of $N_+ - N_-$ in Proposition 3.13, we can ignore the contribution of the $q$-independent term $-\log 2$ in (3.36) and produce our final expression for the term $N_+ - N_-$.

**Proposition 3.14.** The term $N_+ - N_-$ in (3.26) is given by

$$\int_0^1 \frac{d}{dq}\left[\bar{\zeta}(c,q)\left(\bar{\zeta}(a,q)-\bar{\zeta}(a,1-q)\right)\right]\log\sin(\pi q)\,dq$$
$$= -\pi\int_0^1 \bar{\zeta}(c,q)\left[\bar{\zeta}(a,q)-\bar{\zeta}(a,1-q)\right]\cot(\pi q)\,dq.$$

Taking the limit as $\varepsilon \to 0$ in (3.22), we obtain the result described in Theorem 2.1.

## 4. An expression for the Tornheim sum $T(m,0,n)$

In this section we analyze the behavior of $T(a,0,c)$ given in Theorem 2.1 as the parameters $a$ and $c$ approach positive integer values. The notation $a = m+\varepsilon_a,\ c = n+\varepsilon_c$ with $m,n\in\mathbb{N}, m,n\geq 2$ and $\varepsilon_a,\varepsilon_a \to 0$, is empoyed throughout.

Observe that $\lambda(z)$ is singular as $z$ becomes a positive integer, so in order to obtain the limiting value $T(m,0,n)$, it remains to consider the limiting behavior of $T(a,0,c)$.

Define

$$L_1(m,n) := \lim_{a\to m}\lim_{c\to n} 4\lambda(a)\lambda(c)\sin\left(\frac{\pi a}{2}\right)\sin\left(\frac{\pi c}{2}\right) \times\left\{\bar{\zeta}\,(a)\,\bar{\zeta}\,(c) - \frac{\bar{\zeta}\,(a+c)\,B(a,c)}{1-\tan\left(\frac{\pi a}{2}\right)\tan\left(\frac{\pi c}{2}\right)}\right\} \tag{4.1}$$

and

$$L_2(m,n) := -\lim_{a\to m}\lim_{c\to n} 2\lambda(a)\lambda(c)\cos\left(\frac{\pi a}{2}\right)\sin\left(\frac{\pi c}{2}\right) \times\int_0^1\left[\bar{\zeta}(a,q)-\bar{\zeta}(a,1-q)\right]\bar{\zeta}(c,q)\cot\pi q\,dq. \tag{4.2}$$

The expressions for $L_1(m,n)$ and $L_2(m,n)$ given in this section give Theorem 2.2 from Theorem 2.1.

**The evaluation of $L_1(m,n)$.** This is elementary. The simplification employs the relations

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!}, \qquad k\in\mathbb{N}\cup\{0\}, \tag{4.3}$$
$$\zeta(1-k) = \frac{(-1)^{k+1}B_k}{k}, \qquad k\in\mathbb{N},$$

and

$$\zeta'(-2k) = (-1)^k \frac{(2k)!\zeta(2k+1)}{2(2\pi)^{2k}}, \qquad k \in \mathbb{N}.$$

**Proposition 4.1.** The function $L_1(m,n)$ defined in (4.1) is given by

$$L_1(m,n) = \zeta(m)\zeta(n) - \frac{1}{2}\zeta(m+n). \tag{4.4}$$

*Proof.* The expansion of $\lambda(z)$ about $z = k \in \mathbb{N}$ is

$$\lambda(k+\varepsilon) = (-1)^k \frac{(2\pi)^{k-1}}{\Gamma(k)} \left[ \frac{1}{\varepsilon} + \log 2\pi - \psi(k) + O(\varepsilon) \right]. \tag{4.5}$$

We now examine the behavior of the other factors in (4.1) as $\varepsilon_a, \varepsilon_c$ tend to zero. The result depends on the parities of $m$ and $n$.

To simplify our notation we define the 4-*parity* of the integer $k$, $p_k$, as

$$p_k := (-1)^{\lfloor k/2 \rfloor} = \begin{cases} (-1)^{k/2}, & k \text{ even}, \\ (-1)^{(k-1)/2}, & k \text{ odd}, \end{cases} \tag{4.6}$$

Then, for $k$ even:

$$\sin\left(\frac{\pi}{2}(k+\varepsilon)\right) = \frac{\pi}{2} p_k \varepsilon + O\left(\varepsilon^2\right), \tag{4.7}$$

$$\tan\left(\frac{\pi}{2}(k+\varepsilon)\right) = \frac{\pi}{2} \varepsilon + O\left(\varepsilon^2\right), \tag{4.8}$$

$$\bar{\zeta}(k+\varepsilon) = \zeta(1-k) + O(\varepsilon), \tag{4.9}$$

and for $k$ odd:

$$\sin\left(\frac{\pi}{2}(k+\varepsilon)\right) = p_k + O(\varepsilon), \tag{4.10}$$

$$\tan\left(\frac{\pi}{2}(k+\varepsilon)\right) = -\frac{2}{\pi\varepsilon} + O\left(\varepsilon^0\right), \tag{4.11}$$

$$\bar{\zeta}(k+\varepsilon) = -\zeta'(1-k)\varepsilon + O\left(\varepsilon^2\right), \qquad k \geq 3, \tag{4.12}$$

since $\zeta(-2k) = 0$ for $k \in \mathbb{N}$.

Therefore

(a) $m$ and $n$ even:

$$L_1(m,n) = \frac{(2\pi)^{m+n}}{4\Gamma(m)\Gamma(n)} \left[\zeta(1-m)\zeta(1-n) - B(m,n)\zeta(1-m-n)\right] p_m p_n$$

(b) $m$ even and $n$ odd:

$$L_1(m,n) = \frac{(2\pi)^{m+n-1}}{\Gamma(m)\Gamma(n)} \left[\zeta(1-m)\zeta'(1-n) - B(m,n)\zeta'(1-m-n)\right] p_m p_n$$

(c) $m$ odd and $n$ even:

$$L_1(m,n) = \frac{(2\pi)^{m+n-1}}{\Gamma(m)\,\Gamma(n)} \left[\zeta'(1-m)\,\zeta(1-n) - B(m,n)\zeta'(1-m-n)\right] p_m p_n$$

(d) $m$ and $n$ odd:

$$L_1(m,n) = \frac{(2\pi)^{m+n}}{4\Gamma(m)\,\Gamma(n)} \left[\frac{4}{\pi^2}\zeta'(1-m)\,\zeta'(1-n) + B(m,n)\zeta(1-m-n)\right] p_m p_n.$$

The result now follows by using the relations (4.3). □

**The evaluation of $L_2(m,n)$.** This proceeds along similar lines. We employ the expansion

$$\begin{aligned}\bar{\zeta}(k+\varepsilon,q) &= \bar{\zeta}(k,q) + \varepsilon\bar{\zeta}'(k,q) + o(\varepsilon) \\ &= -\frac{1}{k}\left[B_k(q) + \varepsilon A_k(q)\right] + o(\varepsilon),\end{aligned}$$

and the reflection property of the Bernoulli polynomials,

$$B_k(1-q) = (-1)^k B_k(q), \tag{4.13}$$

to establish the vanishing of some of the integrals that appear in intermediate calculations. The results are expressed in terms of the four integrals $I_{AA}$, $I_{AB}$, $I_{BB}$, $J_{AA}$ introduced in (2.5).

**Proposition 4.2.** The limit $L_2(m,n)$ defined in (4.2) is given by

$$L_2(m,n) = p_{m+n}\frac{(2\pi)^{m+n-1}}{m!n!}\ell_2(m,n). \tag{4.14}$$

Here $p_{n+m}$ is defined in (4.6) and the function $\ell_2(m,n)$ is given in terms of the following basic integrals,

$$I_{BB}(k,l) := \int_0^1 B_k(q)B_l(q)K(q)\,dq, \tag{4.15}$$

$$I_{AB}(k,l) := \frac{1}{\pi}\int_0^1 A_k(q)B_l(q)K(q)\,dq, \tag{4.16}$$

$$I_{AA}(k,l) := \frac{1}{\pi^2}\int_0^1 A_k(q)A_l(q)K(q)\,dq, \tag{4.17}$$

$$J_{AA}(k,l) := \frac{1}{\pi^2}\int_0^1 A_k(q)A_l(1-q)K(q)\,dq. \tag{4.18}$$

by

(a) $m$ and $n$ even:

$$\ell_2(m,n) = mI_{AB}(m-1,n) + nI_{AB}(m,n-1), \tag{4.19}$$

(b) $m$ and $n$ odd:

$$\ell_2(m,n) = mI_{AB}(n,m-1) + nI_{AB}(n-1,m). \tag{4.20}$$

(c) $m$ odd and $n$ even:

$$\ell_2(m,n) = \frac{1}{2}\left(mI_{BB}(m-1,n) + nI_{BB}(m,n-1)\right), \tag{4.21}$$

(d) $m$ even and $n$ odd:

$$\begin{aligned}\ell_2(m,n) &= -mI_{AA}(m-1,n) - nI_{AA}(m,n-1) \\ &\quad - mJ_{AA}(m-1,n) + nJ_{AA}(m,n-1),\end{aligned} \tag{4.22}$$

*Proof.* We make use of the expansion

$$\cos\left(\frac{\pi}{2}(k+\varepsilon)\right) \quad = \quad \begin{cases} p_k + o(\varepsilon), & k \text{ even} \\ -p_k\frac{\pi}{2}\varepsilon + o(\varepsilon), & k \text{ odd.}\end{cases} \tag{4.23}$$

In the case $m$ even, the singularity of $\lambda(m+\varepsilon_a)$ as $\varepsilon_a \to 0$ is balanced by

$$\bar{\zeta}\left(m+\varepsilon_a, q\right) - \bar{\zeta}\left(m+\varepsilon_a, 1-q\right) = -\frac{1}{m}\left[A_m(q) - A_m(1-q)\right]\varepsilon_a + O\left({\varepsilon_a}^2\right).$$

For $m$ odd, the combination $\lambda(m+\varepsilon_a)\cos\left((m+\varepsilon_a)\pi/2\right)$ is regular as $\varepsilon_a \to 0$. The singularity at $n+\varepsilon_c$ is treated similarly: the term $\lambda(n+\varepsilon_c)\sin\left((n+\varepsilon_c)\pi/2\right)$ is regular for $n$ even; for $n$ odd, the singularity of $\lambda(n+\varepsilon_c)$ requires the vanishing of the integral in (4.2). This is a consequence of the symmetry of the function $\left[\bar{\zeta}(a,q) - \bar{\zeta}(a,1-q)\right]\cot\pi q$, about $q = 1/2$.

Introduce the notation

$$\alpha_{m,n} := p_{m+n}\frac{(2\pi)^{m+n-2}}{m!\,n!}, \tag{4.24}$$

where $p_{m+n}$ is defined in (4.6). Then

(a) for $m$ and $n$ even:

$$L_2(m,n) = -\pi\alpha_{m,n}\int_0^1 [A_m(q) - A_m(1-q)]B_n(q)\cot\pi q\,dq,$$

(b) for $m$ even and $n$ odd:

$$L_2(m,n) = 2\alpha_{m,n}\int_0^1 [A_m(q) - A_m(1-q)]A_n(q)\cot\pi q\,dq,$$

(c) for $m$ odd and $n$ even:

$$L_2(m,n) = -\pi^2\alpha_{m,n}\int_0^1 B_m(q)B_n(q)\cot\pi q\,dq,$$

(d) for $m$ and $n$ odd:

$$L_2(m,n) = -2\pi\alpha_{m,n}\int_0^1 B_m(q)A_n(q)\cot\pi q\,dq.$$

The final expression in Proposition 4.2 is obtained by writing the integrals above in terms of the basic integrals defined in (4.15). This can be achieved by using

$$\pi\cot\pi q = \frac{d}{dq}\log\sin\pi q = K'(q),$$

integrating by parts and employing the recurrence relations

$$\frac{d}{dq}B_k(q) = kB_{k-1}(q), \tag{4.25}$$

$$\frac{d}{dq}A_k(q) = kA_{k-1}(q) + \frac{k}{k-1}B_{k-1}(q), \quad k \geq 2. \tag{4.26}$$

See [5] for a derivation of (4.26).

The kernel $K(q)$ has only a logarithmic singularity at $q = 0$, so it is possible to separate the integrals involving the difference $A_k(q) - A_k(1-q)$. The functions $A_k(q)$ are well behaved in $[0,1]$. Only $A_1(q) = \log\left[\Gamma(q)/\sqrt{2\pi}\right]$ has a singularity at $q = 0$, and this is only logarithmic, and therefore integrable in the cases of interest here. □

**Example 4.1.** The Tornheim sum $T(2,0,2)$ is the simplest example of even weight. Its exact value is

$$T(2,0,2) = \frac{3}{4}\zeta(4) = \frac{\pi^4}{120}, \tag{4.27}$$

see (6.4). Theorem 2.2 states that $T(2,0,2) = L_1(2,2) + L_2(2,2)$, with

$$L_1(2,2) = \frac{\pi^4}{45}, \tag{4.28}$$

and

$$L_2(2,2) = 4\pi^3\left(I_{AB}(1,2) + I_{AB}(2,1)\right). \tag{4.29}$$

Therefore

$$T(2,0,2) = \frac{\pi^4}{45} + 4\pi^2\int_0^1 A_1(q)B_2(q)K(q)\,dq + 4\pi^2\int_0^1 A_2(q)B_1(q)K(q)\,dq, \tag{4.30}$$

and we have the evaluation

$$\int_0^1 \left[A_1(q)B_2(q) + A_2(q)B_1(q)\right]\log\sin\pi q\,dq = \frac{\pi^2}{288}. \tag{4.31}$$

**Example 4.2.** Theorem 2.2 gives

$$T(2,0,6) = \frac{\pi^8}{8100} + \frac{8\pi^7}{45}I_{AB}(1,6) + \frac{8\pi^7}{15}I_{AB}(2,5). \tag{4.32}$$

**Theorem 4.3.** Every Tornheim sum can be expressed as a finite sum of integrals of the form $I_{AB}$.

## 5. A representation for $I_{AB}(m,n)$ in terms of Clausen functions

In this section we consider the integral

$$I_{AB}(m,n) = \frac{1}{\pi}\int_0^1 A_m(q)B_n(q)K(q)\,dq, \tag{5.1}$$

and express it in terms of the family introduced in (2.11),

$$X_{k,l} := (-1)^{\lfloor l/2 \rfloor}\frac{l!}{(2\pi)^l}\int_0^1 \log\Gamma(q)B_k(q)\,\mathrm{Cl}_{l+1}(2\pi q)\,dq. \tag{5.2}$$

Here $\mathrm{Cl}_l(x)$ are the Clausen functions defined in (1.18) and (1.19).

The reduction of the integrals $I_{AB}$ employs a new family of special functions, $K_n(q)$, $n \in \mathbb{N}_0$, defined as the iterated primitives of the kernel

$$K_0(q) := -K(q) = -\log \sin \pi q, \tag{5.3}$$

that is,

$$K_n(q) := n \int_0^q K_{n-1}(q')dq', \quad n \geq 1. \tag{5.4}$$

The functions $K_n(q)$ are normalized by the condition $K_n(0) = 0$ for $n \geq 1$.

**Note 5.1.** The plan is to use $(n+1)K_n(q) = K'_{n+1}(q)$ and integrate by parts to transform the expression for $I_{AB}(m,n)$ into a finite sum of integrals in which the only type-A function that appears is $A_1(q) = \log \Gamma(q) - \log \sqrt{2\pi}$.

The Fourier expansion of $K_0(q)$ is

$$K_0(q) = \log 2 + \frac{1}{2}\,\mathrm{Cl}_1(2\pi q), \tag{5.5}$$

where $\mathrm{Cl}_1$ is a Clausen function. We first show that $K_n(q)$ is the sum of a polynomial in $q$ of degree $n$ and a multiple of the Clausen function $\mathrm{Cl}_{n+1}(2\pi q)$ (see (5.22)). We then show that the polynomial part can be integrated out explicitly in terms of zeta values, leading finally to the analytic expression for the Tornheim sums $T(m,0,n)$ of even weight $N = m+n$ in terms of the integrals $X_{kl}$, given in Theorem 2.3.

We first recall some basic properties of the functions $A_m(q)$, introduced in [5]. The function

$$A_m(q) := m \frac{\partial}{\partial z}\zeta(z,q)\Big|_{z=1-m} \tag{5.6}$$

satisfies, for $m \in \mathbb{N} - \{1\}$, the recurrence

$$\frac{d}{dq}A_m(q) = mA_{m-1}(q) + \frac{m}{m-1}B_{m-1}(q), \tag{5.7}$$

with initial condition

$$A_1(q) = \log \Gamma(q) - \log \sqrt{2\pi}. \tag{5.8}$$

The boundary values are given by

$$A_m(0) = A_m(1) = m\zeta'(1-m), \text{ for } m \geq 2, \tag{5.9}$$

and

$$A_1(0^+) = -\infty, \qquad A_1(1) = -\log \sqrt{2\pi}. \tag{5.10}$$

The recurrence (5.7) is similar to the relation

$$\frac{d}{dq}B_m(q) = mB_{m-1}(q), \tag{5.11}$$

satisfied by the Bernoulli polynomials.

**Theorem 5.2.** The integral $I_{AB}(m,n)$ can be expressed in terms of the family

$$U_{k,l} := \int_0^1 \log \Gamma(q) B_k(q) K_l(q)\, dq, \tag{5.12}$$

with $k + l = m + n - 1$.

*Proof.* The proof is by induction on $m$. We show first that any integral of the form

$$\int_0^1 A_m(q)B_n(q)K_p(q)\,dq$$

can be expressed in terms of

$$U^*_{k,l} := \int_0^1 A_1(q)B_k(q)K_l(q)\,dq, \tag{5.13}$$

with $k+l+1 = m+n+p$, and then transform into $U_{k,l}$ by using $A_1(q) = \log\Gamma(q) - \log\sqrt{2\pi}$.

Start with

$$(p+1)\int_0^1 A_m(q)B_n(q)K_p(q)\,dq = \int_0^1 A_m(q)B_n(q)\frac{d}{dq}K_{p+1}(q)\,dq.$$

Integrate by parts and use $K_{p+1}(0) = 0$, (5.7), (5.9) and (5.11) to derive

$$\begin{aligned}(p+1)\int_0^1 A_m(q)B_n(q)K_p(q)\,dq = {} & A_m(1)B_n(1)K_{p+1}(1)\\ & - m\int_0^1 A_{m-1}(q)B_n(q)K_{p+1}(q)\,dq\\ & - \frac{m}{m-1}\int_0^1 B_{m-1}(q)B_n(q)K_{p+1}(q)\,dq\\ & - n\int_0^1 A_m(q)B_{n-1}(q)K_{p+1}(q).\end{aligned} \tag{5.14}$$

The term involving the product of two Bernoulli polynomials can be evaluated in closed form, and only the last term in (5.14) requires further analysis. Repeating the process to this last term produces

$$\begin{aligned}(p+1)(p+2)\int_0^1 A_m(q)B_{n-1}(q)K_{p+1}(q)\,dq = A_m(1)B_{n-1}(1)K_{p+2}(1)\\ - m\int_0^1 A_{m-1}(q)B_{n-1}(q)K_{p+2}(q)\,dq - \frac{m}{m-1}\int_0^1 B_{m-1}(q)B_{n-1}(q)K_{p+2}(q)\,dq\\ - (n-1)\int_0^1 A_m(q)B_{n-2}(q)K_{p+2}(q)\,dq.\end{aligned}$$

We conclude that each integration by parts produces terms that can be evaluated in the stated closed form (by the induction hypothesis) and one integral where the index of the Bernoulli polynomial term is decreased by one. Note that the indices $j,k,l$ of the three functions in any term of the integrand appearing in this procedure satisfy $j+k+l = m+n+p$. The case of interest, $I_{AB}(m,n)$, corresponds to $p=0$.

Eventually we arrive at

$$(p+n+1)\int_0^1 A_m(q)B_0(q)K_{p+n}(q)\,dq = A_m(1)B_0(1)K_{p+n+1}(1) - \int_0^1 K_{p+n+1}(q)B_0(q)\left(mA_{m-1}(q)+\frac{m}{m-1}B_{m-1}(q)\right)dq - \int_0^1 A_m(q)\frac{d}{dq}B_0(q)K_{p+n+1}(q)\,dq.$$

The fact that $B_0(q)\equiv 1$, so that the last integral vanishes identically, completes the argument.

It remains to prove that integrals of the form

$$V_{k,m,n} := \int_0^1 B_k(q)B_m(q)K_n(q)\,dq \tag{5.15}$$

can be evaluated in closed form and to use the relation (5.8) to eliminate $A_1(q)$ from the formulas.

The first step in this part of the proof is to use the relation

$$B_{n_1}(q)B_{n_2}(q) = \sum_{k=0}^{k(n_1,n_2)}\left[n_1\binom{n_2}{2k}+n_2\binom{n_1}{2k}\right]\frac{B_{2k}}{n_1+n_2-2k}B_{n_1+n_2-2k}(q) + (-1)^{n_1+1}\frac{n_1!n_2!}{(n_1+n_2)!}B_{n_1+n_2}, \tag{5.16}$$

with

$$k(n_1,n_2) = \text{Max}\,\{\lfloor n_1/2\rfloor\},\lfloor n_2/2\rfloor\}\}, \tag{5.17}$$

for the product of two Bernoulli polynomials in order to reduce $V_{k,m,n}$ to sums of terms of the form

$$W_{m,n} := \int_0^1 B_m(q)K_n(q)\,dq. \tag{5.18}$$

Integration by parts show that (5.18) can be written in terms of the $W$-integrals with second index 0. Indeed,

$$\begin{aligned} W_{m,n} &= \frac{1}{m+1}\int_0^1 K_n(q)\frac{d}{dq}B_{m+1}(q)\,dq \\ &= \frac{B_{m+1}(1)K_n(1)}{m+1} - \frac{1}{m+1}\int_0^1 B_{m+1}(q)\frac{d}{dq}K_n(q) \\ &= \frac{B_{m+1}(1)K_n(1)}{m+1} - \frac{n}{m+1}W_{m+1,n-1}, \end{aligned}$$

and iterating this procedure yields

$$W_{m,n} = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{(m+1)_k}B_{m+k}(1)K_{n-k+1}(1) + \frac{(-1)^n}{(m+1)_n}W_{m+n,0}, \tag{5.19}$$

where $(m)_k = m(m+1)\cdots(m+k-1)$ is the Pochhammer symbol.

The closed form of $V_{k,m,n}$ follows from

$$W_{r,0} = \int_0^1 B_r(q) K_0(q)\, dq = \begin{cases} \log 2 & r = 0, \\ 0 & r \text{ odd}, \\ (-1)^{r/2} r! \zeta(r+1)/(2\pi)^r, & r > 0 \text{ even}, \end{cases} \tag{5.20}$$

given as Example 5.2 in [4]. □

**Note 5.3.** The boundary terms $K_n(1)$ are given in (5.29).

Our final step in the reduction of the integrals $I_{AB}$ makes use of a representation of the kernels $K_n(q)$ in terms of the Clausen functions $\text{Cl}_n(x)$, defined in (1.18) and (1.19).

**Proposition 5.4.** The kernels $K_n(q)$ are given by

$$K_0(q) = \log 2 + \frac{1}{2} \text{Cl}_1(2\pi q), \tag{5.21}$$

and, for $n \geq 1$,

$$K_n(q) = q^n \log 2 + n! \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1} \zeta(2k+1)}{(2\pi)^{2k} (n-2k)!} q^{n-2k} + p_n \frac{n!}{(2\pi)^n} \text{Cl}_{n+1}(2\pi q), \tag{5.22}$$

where the coefficient $p_n = (-1)^{\lfloor n/2 \rfloor}$ is the 4-*parity* of $n$ defined in (4.6).

*Proof.* The Fourier series expansion of the function $K_0(q) = -\log \sin(\pi q)$

$$K_0(q) = \log 2 + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos(2\pi k q)}{k}. \tag{5.23}$$

is standard. To compute the expansion for $K_1(q)$, we cannot just integrate term by term the series (5.23), since it is not uniformly convergent for $q \in [0, 1]$. To bypass this problem, observe that the indefinite integral of $K_0(q)$ can be written as

$$\int K_0(q) dq = q \log\left(1 - e^{2\pi q i}\right) - q \log \sin(\pi q) - \frac{\pi i}{2} q^2 - \frac{i}{2\pi} \text{Li}_2\left(e^{2\pi q i}\right), \tag{5.24}$$

where $\text{Li}_n(z)$ is the polylogarithmic function defined for $z \in \mathbb{C}, |z| \leq 1$ by

$$\text{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad n \geq 2. \tag{5.25}$$

Separating the real and imaginary parts we obtain, for $0 \leq q \leq 1$,

$$\int K_0(q) dq = q \log 2 + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k q)}{k^2} + i \left[ \frac{\pi}{2} q(1-q) + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\cos(2\pi k q)}{k^2} \right], \tag{5.26}$$

and the proposition is proved for $n = 1$ since the term in brackets is a $q$-independent constant. This can be seen directly from Hurwitz's Fourier series representation of the Bernoulli polynomials

$$B_n(q) = -\frac{n!}{(2\pi i)^n} \sum_{k=-\infty}^{\infty}{}' \frac{e^{2\pi i k q}}{k^n}, \tag{5.27}$$

where the prime indicates that the term $k=0$ must be excluded in the sum.

The result for $n \geq 2$ can be obtained directly from the expression for $K_1(q)$,

$$K_1(q) = q\log 2 + \frac{1}{2\pi}\sum_{k=1}^{\infty} \frac{\sin(2\pi k q)}{k^2}, \tag{5.28}$$

integrating successively term by term, which is now legitimate since, for $n \geq 2$, the series defining the Clausen function $\text{Cl}_n(x)$ is uniformly convergent. □

**Note 5.5.** The evaluation of the basic integral $I_{AB}(m,n)$ requires the boundary values of $K_n(q)$, $n \in \mathbb{N}$, at $q=1$. These are given by

$$K_n(1) = \log 2 + n! \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^{k+1}\zeta(2k+1)}{(2\pi)^{2k}(n-2k)!}, \qquad n \in \mathbb{N}, \tag{5.29}$$

where we have used the special values $\text{Cl}_{2n}(2\pi) = 0$ and $\text{Cl}_{2n+1}(2\pi) = \zeta(2n+1)$.

**Note 5.6.** The method presented in this section reduces the evaluation of all *even*-weight Tornheim sums to the evaluation of the integrals $U_{m,n}$, defined in (5.12). The expansion (5.22) and the formula

$$\begin{aligned} \int_0^1 q^n \log\Gamma(q)\,dq = {} & \frac{1}{n+1}\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n+1}{2k-1} \frac{(2k)!}{k(2\pi)^{2k}} \left[\delta\zeta(2k) - \zeta'(2k)\right] \\ & - \frac{1}{n+1}\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+1}{2k} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) + \frac{\log\sqrt{2\pi}}{n+1}, \end{aligned} \tag{5.30}$$

(with $\delta = 2\log\sqrt{2\pi} + \gamma$, where $\gamma$ is Euler constant) given as (6.14) in [4], reduces the evaluation of Tornheim sums to the evaluation of the integrals $X_{m,n}$, introduced in (5.2). The result described in Theorem 2.3 appears from transforming the explicit representation (2.6) in Theorem 2.2 by the methods presented in this section. The details are left to the reader.

**Note 5.7.** The value

$$X_{0,n} = (-1)^{\lfloor n/2 \rfloor} \frac{n!}{(2\pi)^n} \int_0^1 \log\Gamma(q)\,\text{Cl}_{n+1}(2\pi q)\,dq \tag{5.31}$$

can be obtained from the integrals

$$\int_0^1 \log\Gamma(q) \sin(2\pi k q)\,dq = \frac{A + \log k}{2\pi k},$$

$$\int_0^1 \log\Gamma(q) \cos(2\pi k q)\,dq = \frac{1}{4k},$$

where $A = \log(2\pi) + \gamma$, with $\gamma$ the Euler constant. This appears in [8] 6.443.1 and 6.443.3. It follows that

$$X_{0,n} = (-1)^{\lfloor n/2 \rfloor} \frac{n!}{(2\pi)^n} \times \begin{cases} \frac{1}{2\pi}\left(A\zeta(n+2) - \zeta'(n+1)\right), & \text{for } n \text{ odd}, \\ \frac{1}{4}\zeta(n+2), & \text{for } n \text{ even}. \end{cases} \tag{5.32}$$

For our evaluations of even-weight Tornheim sums, we will only need the result for $n$ even.

**Example 5.8.** This is a continuation of Example 4.1. We express here the sum $T(2,0,2)$ first in terms of the $U$-integrals. Theorem 2.2, with $m=n=2$, gives

$$T(2,0,2) = \zeta(2)^2 - \frac{1}{2}\zeta(4) - 4\pi^2 \left[ \int_0^1 A_1(q)B_2(q)K_0(q)dq \right.$$
$$\left. + \int_0^1 A_2(q)B_1(q)K_0(q)dq \right].$$

The second integral is now modified to avoid the presence of the function $A_2(q)$. Write $K_0(q) = (d/dq)K_1(q)$, integrate by parts and use formulas (5.7) and (5.11) to get

$$\int_0^1 A_2(q)B_1(q)K_0(q)dq = A_2(1)B_1(1)K_1(1) - \int_0^1 A_2(q)B_0(q)K_1(q)dq$$
$$- 2\int_0^1 A_1(q)B_1(q)K_1(q)dq - 2\int_0^1 B_1^2(q)K_1(q)dq.$$

The new integral involving $A_2(q)B_0(q)K_1(q)$ can be dealt with in a similar manner, to produce

$$\int_0^1 A_2(q)B_1(q)K_0(q)\,dq = A_2(1)B_1(1)K_1(1) - \frac{1}{2}A_2(1)B_0(1)K_2(1)$$
$$+ \int_0^1 A_1(q)B_0(q)K_2(q)\,dq - 2\int_0^1 A_1(q)B_1(q)K_1(q)\,dq$$
$$+ \int_0^1 B_0(q)B_1(q)K_2(q)\,dq - 2\int_0^1 B_1^2(q)K_1(q)\,dq.$$

From (5.9) and (5.29) we see that the boundary terms calcel out and therefore the Tornheim sum $T(2,0,2)$ is given by

$$T(2,0,2) = \zeta^2(2) - \zeta(4)/2 - 4\pi^2\left(U^*_{2,0} - 2U^*_{1,1} + U^*_{0,2} + V_{0,1,2} - 2V_{1,1,1}\right).$$

The expression for $V_{k,m,n}$ shows that $V_{0,1,2} = 2V_{1,1,1} = \frac{1}{12}\log 2$. Therefore, the sum $T(2,0,2)$ is expressed in terms of the $U^*$-integrals as

$$T(2,0,2) = \frac{\pi^4}{45} - 4\pi^2\left(U^*_{2,0} - 2U^*_{1,1} + U^*_{0,2}\right).$$

The transformation from $U^*_{m,n}$ to $U_{m,n}$ gives

$$T(2,0,2) = \frac{\pi^4}{45} + \frac{1}{3}\log(2\pi)\left(\pi^2\log(2) + 9\zeta(3)\right) - 4\pi^2\left(U_{2,0} - 2U_{1,1} + U_{0,2}\right).$$

The final transformation is to write $U_{m,n}$ in terms of $X_{m,n}$ to obtain

$$T(2,0,2) = \frac{\pi^4}{45} - Y^*_{2,2}, \tag{5.33}$$

where

$$Y^*_{2,2} := 4\pi^2\left(X_{0,2} - 2X_{1,1} + X_{2,0}\right) - 2\zeta(3)\log(2\pi). \tag{5.34}$$

**Example 5.9.** At weight 8, we consider first the value of

$$T(2,0,6) = \frac{\pi^8}{8100} + \frac{4\pi^7}{45}\left(2I_{AB}(1,6) + 6I_{AB}(2,5)\right).$$

Expressing the $I_{AB}$ integrals in terms of the $U^*$ integrals we find

$$2I_{AB}(1,6)+6I_{AB}(2,5)= \\ -\frac{2}{\pi}\left(U^*_{0,6}-6U^*_{1,5}+15U^*_{2,4}-20U^*_{3,3}+15U^*_{4,2}-6U^*_{5,1}+U^*_{6,0}\right),$$

and in terms of the $U$ integrals:

$$2I_{AB}(1,6)+6I_{AB}(2,5)=\frac{1}{2\pi}\log(2\pi)\left(\frac{\log(2)}{21}-\frac{\zeta(3)}{2\pi^2}-\frac{15\zeta(5)}{2\pi^4}+\frac{315\zeta(7)}{2\pi^6}\right) \\ -\frac{2}{\pi}\left(U_{0,6}-6U_{1,5}+15U_{2,4}-20U_{3,3}+15U_{4,2}-6U_{5,1}+U_{6,0}\right).$$

Thus we obtain the representation

$$T(2,0,6)=\frac{\pi^8}{8100}+\frac{2\pi^6}{45}\log(2\pi)\left(\frac{\log(2)}{21}-\frac{\zeta(3)}{2\pi^2}-\frac{15\zeta(5)}{2\pi^4}+\frac{315\zeta(7)}{2\pi^6}\right) \\ -\frac{8\pi^6}{45}\left(U_{0,6}-6U_{1,5}+15U_{2,4}-20U_{3,3}+15U_{4,2}-6U_{5,1}+U_{6,0}\right). \tag{5.35}$$

In terms of the $X_{m,n}$ integrals we have

$$T(2,0,6)=\frac{\pi^8}{8100}-Y^*_{6,2}, \tag{5.36}$$

where

$$Y^*_{6,2}:=\frac{8\pi^6}{45}\left(X_{0,6}-6X_{1,5}+15X_{2,4}-20X_{3,3}+15X_{4,2}-6X_{5,1}+X_{6,0}\right) \\ -6\zeta(7)\log(2\pi). \tag{5.37}$$

**Example 5.10.** The other two Tornheim sums of weight 8 are $T(3,0,5)$ and $T(4,0,4)$. They are given by

$$T(3,0,5)=-\frac{\pi^8}{18900}+\zeta(3)\zeta(5)+\frac{8\pi^7}{45}\left(5I_{AB}(4,3)+3I_{AB}(5,2)\right)$$

and

$$T(4,0,4)=\frac{\pi^8}{14175}+\frac{8\pi^7}{9}\left(I_{AB}(3,4)+I_{AB}(4,3)\right),$$

respectively.

Expressing the $I_{AB}$ integrals in terms of the $X$ integrals we find

$$T(3,0,5)=-\frac{\pi^8}{18900}-6\zeta(3)\zeta(5)-Y^*_{3,5}, \tag{5.38}$$

where

$$Y^*_{3,5}:=\frac{32\pi^6}{9}\left(X_{0,6}-3X_{1,5}+3X_{2,4}-X_{3,3}\right)-15\zeta(7)\log(2\pi), \tag{5.39}$$

and

$$T(4,0,4)=\frac{\pi^8}{14175}-4\zeta(3)\zeta(5)-Y^*_{4,4}, \tag{5.40}$$

where

$$Y^*_{4,4} := \frac{8\pi^6}{3}\left(X_{0,6} - 4X_{1,5} + 6X_{2,4} - 4X_{3,3} + X_{4,2}\right) - 20\zeta(7)\log(2\pi). \tag{5.41}$$

**Note 5.11.** In the examples given above, the integrals $X_{k,l}$ appear in a symmetric form. This is a general feature, as stated in Theorem 2.3. Depending on the parity of the integers $m$ and $n$, the even-weight Tornheim sum $T(m,0,n)$ contains either $Y_{m,n}$ or $Y_{n,m}$, where

$$Y_{m,n} := \frac{2(2\pi)^{m+n-2}}{m!(n-2)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}X_{j,m+n-2-j}.$$

There are linear relations among the integral $Y_{m,n}$ of fixed weight $N := m+n$. These come from the linear relations among the Tornheim sums $T(m,0,n)$ of the same weight, discussed in the next section. For example, the identity

$$T(2,0,6) + T(6,0,2) = \frac{2}{3}\zeta(8)$$

gives

$$Y_{2,6} + Y_{6,2} = 12\log(2\pi)\zeta(7) + \frac{7}{3}\zeta(8) - 6\zeta(3)\zeta(5),$$

whereas

$$5T(6,0,2) + 2T(5,0,3) = \frac{163}{12}\zeta(8) - 8\zeta(3)\zeta(5)$$

gives

$$5Y_{2,6} + 2Y_{5,3} = 60\log(2\pi)\zeta(7) + \frac{29}{6}\zeta(8) - 30\zeta(3)\zeta(5).$$

A systematic study of the integrals $Y_{m,n}$ will be presented elsewhere.

## 6. A systematic list of examples

Here we present a systematic evaluation of the Tornheim sums $T(m,k,n)$ with $m,\ k,\ n \in \mathbb{N}\cup\{0\}$. The sums are organized according to the weight $N = m+k+n$. The conditions $m+n \geq 2$, $k+n \geq 2$ and $N \geq 3$ are imposed for convergence. The symmetry relation $T(m,k,n) = T(k,m,n)$ is used to impose $m \geq k$.

We use the notation $\mathcal{Z}_N$ and $\mathcal{Z}_N^0$ introduced in the introduction. Huard's result (1.4) allows us to evaluate any Tornheim sum in $\mathcal{Z}_N$ in terms of the sums in $\mathcal{Z}_N^0$. Before detailing a systematic algorithm to evaluate all the sums in $\mathcal{Z}_N$ for a given weight $N$ and giving specific examples, we shall determine how many of the Tornheim sums in $\mathcal{Z}_N^0$, for a given *even* weight $N$, remain undetermined after using all the known algebraic identities for these sums.

As we discussed in the introduction, two of the sums in $\mathcal{Z}_N^0$ have known explicit evaluations:

$$T(0,0,N) = \zeta(N-1) - \zeta(N), \quad N \geq 3, \tag{6.1}$$

and

$$T(1,0,N-1) = \frac{1}{2}\left[(N-1)\zeta(N) - \sum_{i=2}^{N-2}\zeta(i)\zeta(N-i)\right], \quad N \geq 3. \tag{6.2}$$

Euler proved that for $m \geq 2, n \geq 2$ the sums $T(m,0,n)$ satisfy the symmetrized identity

$$T(m,0,n) + T(n,0,m) = \zeta(m)\zeta(n) - \zeta(m+n), \tag{6.3}$$

so that only the case $m \geq n$ needs to be considered. In particular, for $m = n$ we find

$$T(n,0,n) = \frac{1}{2}\zeta^2(n) - \frac{1}{2}\zeta(2n). \tag{6.4}$$

At even weight $N$, the previous identities leave exactly $N/2-2$ Tornheim sums of the type $T(m,0,n)$ unevaluated. In effect, the convergence conditions require $n \geq 2$, so that all the sums with $m = N/2+1, N/2+2, \ldots, N-2$ are undetermined. The remaining $N/2-2$ unevaluated Tornheim sums are not all independent, since they satisfy linear relations obtained by applying Huard's identity (1.4) to some special cases with known evaluations as, for instance,

$$T(m,k,0) = \zeta(m)\zeta(k), \tag{6.5}$$

$$T(m,k,1) = (-1)^m \left\{ \sum_{i=2}^{m} (-1)^i \zeta(i)\zeta(N-i) + \frac{1}{2}\sum_{i=2}^{N-2} \zeta(i)\zeta(N-i) - \frac{1}{2}(N+1)\zeta(N) \right\}, \tag{6.6}$$

or

$$T(m,1,1) = \frac{1}{2}\left( (N+1)\zeta(N) - \sum_{i=2}^{N-2} \zeta(i)\zeta(N-i) \right). \tag{6.7}$$

These results appear in [11]. As usual, $N$ is the weight of the Tornheim sum in the left hand side.

On the other hand, Granville [9] showed that the multiple zeta values defined in (1.8) satisfy

$$\sum \zeta(p_1, p_2, \cdots, p_g) = \zeta(N), \tag{6.8}$$

where the sum is over all elements of $\mathbb{Z}^g$ such that $p_1 + p_2 + \cdots + p_g = N$, with $p_j \geq 1$ and $p_1 \geq 2$. In particular, when $g = 2$ we obtain

$$\sum_{m+n=N} T(m,0,n) = \zeta(N), \tag{6.9}$$

where the sum is over pairs $(m,n)$ with $m \geq 1$ and $n \geq 2$.

**Experimental observation**. Not all of the relations stated above are linearly independent. However, it is possible to show that all Tornheim sums in $\mathcal{Z}_N^0$ for weights $N = 4$ and $N = 6$ can be completely evaluated in terms of zeta values (the details are provided at the end of this section). In the case of even weight $N \geq 8$ the Tornheim sums can be expressed in terms of zeta values and a reduced number of basis sums, which we choose as $T(N-2k, 0, 2k)$, with $k = 1, \ldots, K$, where $K$ is given by

$$K = \left\lfloor \frac{N-2}{6} \right\rfloor \tag{6.10}$$

In particular, we need only one basis sum for weights 8, 10 and 12; two for weights 14, 16 and 18, and three for weights 20, 22 and 24. The reader will find in [1] that $K$ given above is an upper bound for the number of Tornheim sums required.

**Example 6.1.** Indeed, for weight $N = 8$ we find that all sums in $\mathcal{Z}_8^0$ have either explicit evaluations or can be expressed in terms of just $T(6,0,2)$:

$$\begin{aligned}
T(0,0,8) &= \zeta(7) - \zeta(8), \\
T(1,0,7) &= \tfrac{5}{4}\zeta(8) - \zeta(3)\zeta(5), \\
T(2,0,6) &= \tfrac{2}{3}\zeta(8) - T(6,0,2), \\
T(3,0,5) &= -\tfrac{187}{24}\zeta(8) + 5\zeta(3)\zeta(5) + \tfrac{5}{2}T(6,0,2), \\
T(4,0,4) &= \tfrac{1}{12}\zeta(8), \\
T(5,0,3) &= \tfrac{163}{24}\zeta(8) - 4\zeta(3)\zeta(5) - \tfrac{5}{2}T(6,0,2).
\end{aligned}$$

Note that the identity (1.14) follows directly from these expressions.

**Example 6.2.** For weight $N = 14$ we can express all Tornheim sums as zeta values plus the sums $T(12,0,2)$ and $T(10,0,4)$:

$$\begin{aligned}
T(0,0,14) &= \zeta(13) - \zeta(14), \\
T(1,0,13) &= \tfrac{11}{4}\zeta(14) - \zeta(3)\zeta(11) - \zeta(5)\zeta(9) - \tfrac{1}{2}\zeta(7)^2, \\
T(2,0,12) &= \tfrac{271}{420}\zeta(14) - T(12,0,2), \\
T(3,0,11) &= -\tfrac{35741}{840}\zeta(14) + 11\zeta(3)\zeta(11) + 16\zeta(5)\zeta(9) + 9\zeta(7)^2 + \tfrac{11}{2}T(12,0,2), \\
T(4,0,10) &= \tfrac{1}{12}\zeta(14) - T(10,0,4), \\
T(5,0,9) &= \tfrac{40977}{112}\zeta(14) - \tfrac{165}{2}\zeta(3)\zeta(11) - 147\zeta(5)\zeta(9) - \tfrac{345}{4}\zeta(7)^2 \\
&\qquad + \tfrac{9}{2}T(10,0,4) - \tfrac{165}{4}T(12,0,2), \\
T(6,0,8) &= -\tfrac{20773}{35}\zeta(14) + 132\zeta(3)\zeta(11) + 240\zeta(5)\zeta(9) + 141\zeta(7)^2 \\
&\qquad - 6T(10,0,4) + 66T(12,0,2), \\
T(7,0,7) &= \tfrac{1}{2}\zeta(7)^2 - \tfrac{1}{2}\zeta(14), \\
T(8,0,6) &= \tfrac{16619}{28}\zeta(14) - 132\zeta(3)\zeta(11) - 240\zeta(5)\zeta(9) - 141\zeta(7)^2 \\
&\qquad + 6T(10,0,4) - 66T(12,0,2), \\
T(9,0,5) &= -\tfrac{41089}{112}\zeta(14) + \tfrac{165}{2}\zeta(3)\zeta(11) + 148\zeta(5)\zeta(9) + \tfrac{345}{4}\zeta(7)^2 \\
&\qquad - \tfrac{9}{2}T(10,0,4) + \tfrac{165}{4}T(12,0,2), \\
T(11,0,3) &= \tfrac{34901}{840}\zeta(14) - 10\zeta(3)\zeta(11) - 16\zeta(5)\zeta(9) - 9\zeta(7)^2 \\
&\qquad - \tfrac{11}{2}T(12,0,2).
\end{aligned}$$

We are now in a position to formulate a systematic and exhaustive algorithm to evaluate all the Tornheim sums in $\mathcal{Z}_N$. The reader is invited to download the Tornheim Mathematica 6.0 package developed by the authors, available at `http://www.math.tulane.edu/~vhm/packages.html`. Most of the calculations in this paper can be easily reproduced with the aid of this package.

**Algorithm**. The process of determining the Tornheim sums $T(m,k,n)$ proceeds as follows.

**Step 1**. First catalogue all sums that have known explicit evaluations. All of these cases appear in [11]. In this category we find:

The sums with third entry $n=0$:

$$T(m,k,0)=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{r^m\,s^k}=\zeta(m)\zeta(k). \tag{6.11}$$

The sums with third entry $n=1$:

$$\begin{aligned}T(m,k,1)=(-1)^m\Bigg\{&\sum_{i=2}^{m}(-1)^i\zeta(i)\zeta(N-i)+\\ &\frac{1}{2}\sum_{i=2}^{N-2}\zeta(i)\zeta(N-i)-\frac{1}{2}(N+1)\zeta(N)\Bigg\},\end{aligned} \tag{6.12}$$

where $N=m+k+1$ is the weight.

The sums of the type

$$T(1,1,n)=(n+1)\zeta(n+2)-\sum_{i=2}^{n}\zeta(i)\zeta(n+2-i). \tag{6.13}$$

The following sums in $\mathcal{Z}_N^0$:

$$T(0,0,n)=\zeta(n-1)-\zeta(n), \tag{6.14}$$

$$T(1,0,n)=\frac{1}{2}\left(n\zeta(n+1)-\sum_{i=2}^{n-1}\zeta(i)\zeta(n+1-i)\right), \tag{6.15}$$

and the symmetric sum,

$$T(m,0,m)=\frac{1}{2}\zeta^2(m)-\frac{1}{2}\zeta(2m),\qquad m\geq 2. \tag{6.16}$$

For weight $N=m+n$ odd, the sum $T(m,0,n)$ is given by

$$\begin{aligned}T(m,0,n)=(-1)^m&\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{N-2j-1}{m-1}\zeta(2j)\zeta(N-2j)\\ &+(-1)^m\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{N-2j-1}{n-1}\zeta(2j)\zeta(N-2j)-\tfrac{1}{2}\zeta(N).\end{aligned} \tag{6.17}$$

**Step 2**. If $k\neq 0$, use the reduction of Huard et al. given in (1.4) to reduce the $T(m,k,n)$ not covered by the previous step to a finite sum of Tornheim sums in $\mathcal{Z}_N^0$:

(6.18)
$$T(m,k,n) = \sum_{i=1}^{m} \binom{m+k-i-1}{m-i} T(i,0,N-i) + \sum_{i=1}^{k} \binom{m+k-i-1}{k-i} T(i,0,N-i),$$
with $N = m+k+n$.

**Step 3**. For $N = m+n \leq 6$ even and $m,\, n \geq 2$, compute all the sums $T(m,0,n)$ explicitly by solving the set of simultaneous equations obtained from: (a) applying Huard's reduction of the previous step to the identity (6.11), for all independent pairs $(m,k)$ with $m+k=N$; (b) Euler's identity (6.3); and using the known explicit evaluations of the sums in $\mathcal{Z}_N^0$ given in Step 1.

**Step 4**. For $N = m+n \geq 8$ even and $m,\, n \geq 2$, we write all the sums $T(m,0,n)$ in terms of the irreducible basis for weight $N$,

$$\left\{ T(N-2k,0,2k), \quad k = 1, \ldots, \left\lfloor \frac{N-2}{6} \right\rfloor \right\}. \tag{6.19}$$

**Step 5**. The irreducible sums of the previous step are evaluated in terms of the integrals $X_{k,l}$ using Theorem 2.3.

For a given *even* weight $N = m+k+n$, the whole process gives $T(m,k,n)$ as a finite sum (with rational coefficients) of the zeta values $\zeta(N)$ and $\zeta(N-1)$, products of two zeta values of the form $\zeta(j)\zeta(N-j)$ with $2 \leq j \leq N-2$ and, for $N \geq 8$, a finite number of integrals the type $Y^*_{2r,N-2r}$, where

$$Y^*_{2r,N-2r} := Y_{2r,N-2r} + (-1)^{\frac{N}{2}-1} \binom{N-2}{2r-1} \zeta(N-1) \log 2\pi. \tag{6.20}$$

**Definition 6.1.** We say that $(m,k,n)$ is an *admissible triple* if $m \geq k$ and $m,k,n$ satisfy the conditions $m+n \geq 2$, $k+n \geq 2$ and $N \geq 3$ for the convergence of $T(m,k,n)$.

We present now the results for small weight $N = m+k+n$. The cases of weight 3 and 4 are straightforward, as all admissible triples corresponds to cases where the Tornheim sum has an explicit formula.

## **Weight** 3

The admissible triples are $(0,0,3)$, $(1,0,2)$ and $(1,1,1)$. We obtain

$$\begin{aligned} T(0,0,3) &= \zeta(2) - \zeta(3), \\ T(1,0,2) &= \zeta(3), \\ T(1,1,1) &= 2\zeta(3), \end{aligned}$$

directly from (6.1), (6.2) and (6.13), respectively. The reader will find in [3] many proofs of the last identity.

## **Weight** 4

The six admissible triples are

$$(0,0,4),\ (1,0,3),\ (1,1,2),\ (2,0,2),\ (2,1,1) \text{ and } (2,2,0).$$

We obtain

$$\begin{aligned}
T(0,0,4) &= \zeta(3) - \zeta(4) = \zeta(3) - \frac{\pi^4}{90}, \\
T(1,0,3) &= \frac{1}{2}\left(3\zeta(4) - \zeta(2)^2\right) = \frac{1}{4}\zeta(4) = \frac{\pi^4}{360}, \\
T(1,1,2) &= 3\zeta(4) - \zeta(2)^2 = \frac{1}{2}\zeta(4) = \frac{\pi^4}{180}, \\
T(2,0,2) &= \frac{1}{2}\left(\zeta(2)^2 - \zeta(4)\right) = \frac{3}{4}\zeta(4) = \frac{\pi^4}{120}, \\
T(2,1,1) &= \frac{1}{2}\left(5\zeta(4) - \zeta(2)^2\right) = \frac{5}{4}\zeta(4) = \frac{\pi^4}{72}, \\
T(2,2,0) &= \zeta(2)^2 = \frac{5}{2}\zeta(4) = \frac{\pi^4}{36},
\end{aligned}$$

directly from (6.1), (6.2), (6.13), (6.16), (6.12) and (6.11), respectively.

The methods developed here produce the result

$$T(2,0,2) = \frac{\pi^4}{45} + 2\log(2\pi)\zeta(3) - Y_{2,2}. \tag{6.21}$$

It follows that

$$Y_{2,2} = \frac{\pi^4}{72} + 2\log(2\pi)\zeta(3). \tag{6.22}$$

As a consequence of this, we obtain the definite integral

$$\int_0^1 \left(2\pi^2 B_2(q)\,\mathrm{Cl}_1(2\pi q) - 2\pi B_1(q)\,\mathrm{Cl}_2(2\pi q) - B_0(q)\,\mathrm{Cl}_3(2\pi q)\right)\log\Gamma(q)\,dq = \frac{\pi^4}{144} + \log(2\pi)\zeta(3),$$

where

$$B_0(q) = 1, \quad B_1(q) = q - 1/2, \quad B_2(q) = q^2 - q + 1/6,$$

and

$$\mathrm{Cl}_1(2\pi q) = \sum_{k=1}^{\infty} \frac{\cos 2\pi k q}{k}, \quad \mathrm{Cl}_2(2\pi q) = \sum_{k=1}^{\infty} \frac{\sin 2\pi k q}{k^2}, \quad \mathrm{Cl}_3(2\pi q) = \sum_{k=1}^{\infty} \frac{\cos 2\pi k q}{k^3}.$$

Considering that from (5.32) we also know that

$$\int_0^1 \log\Gamma(q)\,\mathrm{Cl}_3(2\pi q)\,dq = \frac{1}{4}\zeta(4) = \frac{\pi^4}{360}, \tag{6.23}$$

we also find

$$\int_0^1 \left(2\pi^2 B_2(q)\,\mathrm{Cl}_1(2\pi q) - 2\pi B_1(q)\,\mathrm{Cl}_2(2\pi q)\right)\log\Gamma(q)\,dq = \frac{7\pi^4}{720} + \log(2\pi)\zeta(3). \tag{6.24}$$

*Not your average integral.*

We state next the values of the Tornheim sums $T(m,k,n)$ of even weight $N \geq 6$. The formulas are the direct output of the algorithms presented here, the only reductions used are those given at the beginning of this section. The reader will observe that the final expresions contain a single even value of the Riemann zeta function. This is artificial. For instance, the value

$$T(4,2,0) = \tfrac{7}{4}\zeta(6),$$

should be written as

$$T(4,2,0) = \zeta(4)\zeta(2),$$

as in (6.11). This latter representation would be more helpful in the search for a closed-form expression for the Tornheim sums. However, at this point, we have decided to minimize the number of zeta values appearing in the formulas.

**Note 6.2.** In the examples that follow, we do not list the Tornheim sum $T(0,0,N)$. This is the only sum that explicitly contains the term $\zeta(N-1)$.

### **Weight** 6

In this example we give complete details, which will be omitted for higher weights. The admissible triples are now $(0,0,6)$, $(1,0,5)$, $(1,1,4)$, $(2,0,4)$, $(2,1,3)$, $(2,2,2)$, $(3,0,3)$, $(3,1,2)$, $(3,2,1)$, $(3,3,0)$, $(4,0,2)$, $(4,1,1)$ and $(4,2,0)$.

A direct application of the identities and explicit formulas already discussed give the following evaluations (in the formulas below we have replaced the product $\zeta(2)\zeta(4)$ by $\frac{7}{4}\zeta(6)$):

$$\begin{aligned}
T(1,0,5) &= \tfrac{3}{4}\zeta(6) - \tfrac{1}{2}\zeta^2(3), & T(1,1,4) &= \tfrac{3}{2}\zeta(6) - \zeta^2(3),\\
T(3,0,3) &= -\tfrac{1}{2}\zeta(6) + \tfrac{1}{2}\zeta^2(3), & T(3,3,0) &= \zeta^2(3),\\
T(4,1,1) &= \tfrac{7}{4}\zeta(6) - \tfrac{1}{2}\zeta^2(3), & T(4,2,0) &= \tfrac{7}{4}\zeta(6).
\end{aligned}$$

Huard's expansion (1.4) gives

$$\begin{aligned}
T(2,1,3) &= 2T(1,0,5) + T(2,0,4),\\
T(2,2,2) &= 4T(1,0,5) + 2T(2,0,4),\\
T(3,1,2) &= 2T(1,0,5) + T(2,0,4) + T(3,0,3),\\
T(3,2,1) &= 6T(1,0,5) + 3T(2,0,4) + T(3,0,3).
\end{aligned}$$

Euler's identity (6.3),

$$T(2,0,4) + T(4,0,2) = \zeta(2)\zeta(4) - \zeta(6) = \tfrac{3}{4}\zeta(6),$$

allows us to express $T(2,0,4)$ in terms of $T(4,0,2)$. This, together with the explicit evaluations of $T(1,0,5)$ and $T(3,0,3)$, yields

$$\begin{aligned}
T(2,0,4) &= \tfrac{3}{4}\zeta(6) - T(4,0,2),\\
T(2,1,3) &= \tfrac{9}{4}\zeta(6) - \zeta^2(3) - T(4,0,2),\\
T(2,2,2) &= \tfrac{9}{2}\zeta(6) - 2\zeta^2(3) - 2T(4,0,2),\\
T(3,1,2) &= \tfrac{5}{2}\zeta(6) - \tfrac{1}{2}\zeta^2(3) - 2T(4,0,2),\\
T(3,2,1) &= 7\zeta(6) - \tfrac{5}{2}\zeta^2(3) - 4T(4,0,2).
\end{aligned}$$

Finally, Huard's expansion (1.4) applied to the Tornheim sum $T(4,2,0) = \frac{7}{4}\zeta(6)$ produces another identity,

$$8T(1,0,5) + 4T(2,0,4) + 2T(3,0,3) + T(4,0,2) = \tfrac{7}{4}\zeta(6),$$

which permits to solve for the as yet undetermined value of $T(4,0,2)$:

$$T(4,0,2) = \tfrac{25}{12}\zeta(6) - \zeta^2(3). \tag{6.25}$$

This last result produces the explicit evaluation of all Tornheim sums of weight 6:

$$\begin{aligned}
&T(1,0,5) = \tfrac{3}{4}\zeta(6) - \tfrac{1}{2}\zeta^2(3), && T(1,1,4) = \tfrac{3}{2}\zeta(6) - \zeta^2(3),\\
&T(2,0,4) = -\tfrac{4}{3}\zeta(6) + \zeta^2(3), && T(2,1,3) = \tfrac{1}{6}\zeta(6),\\
&T(2,2,2) = \tfrac{1}{3}\zeta(6), && T(3,0,3) = -\tfrac{1}{2}\zeta(6) + \tfrac{1}{2}\zeta^2(3),\\
&T(3,1,2) = -\tfrac{1}{3}\zeta(6) + \tfrac{1}{2}\zeta^2(3), && T(3,2,1) = \tfrac{1}{2}\zeta^2(3),\\
&T(3,3,0) = \zeta^2(3), && T(4,0,2) = \tfrac{25}{12}\zeta(6) - \zeta^2(3),\\
&T(4,1,1) = \tfrac{7}{4}\zeta(6) - \tfrac{1}{2}\zeta^2(3), && T(4,2,0) = \tfrac{7}{4}\zeta(6).
\end{aligned}$$

All the Tornheim sums of weight 6 have been evaluated.

**Note 6.3.** The problem of whether these sums are completely reduced is now equivalent to whether $\zeta^2(3)$ and $\zeta(6) = \pi^6/945$ are rationally related. It is conjectured that $\zeta(3)/\pi^3$ is a transcendental number[2].

## Weight 8

The reduction algorithm described above begins by applying Huard's reduction procedure to express every sum $T(m,k,n)$, with $N = m+k+n = 8$, in terms of the $N-2 = 6$ sums

$$\{T(1,0,7),\, T(2,0,6),\, T(3,0,5),\, T(4,0,4),\, T(5,0,3),\, T(6,0,2)\} \tag{6.26}$$

with $k = 0$. For example,

$$T(5,2,1) = 10T(1,0,7) + 5T(2,0,6) + 3T(3,0,5) + 2T(4,0,4) + T(5,0,3). \tag{6.27}$$

We use the shorthand notation $T(5,2,1) = [10,5,3,2,1,0]$. The table below gives all the coefficients corresponding to the 15 Tornheim sums of weight 8 with $k \neq 0$.

$$\begin{aligned}
&T(1,1,6) = [2,0,0,0,0,0], && T(2,1,5) = [2,1,0,0,0,0],\\
&T(2,2,4) = [4,2,0,0,0,0], && T(3,1,4) = [2,1,1,0,0,0],\\
&T(3,2,3) = [6,3,1,0,0,0], && T(3,3,2) = [12,6,2,0,0,0],\\
&T(4,1,3) = [2,1,1,1,0,0], && T(4,2,2) = [8,4,2,1,0,0],\\
&T(4,3,1) = [20,10,4,1,0,0], && T(4,4,0) = [40,20,8,2,0,0],\\
&T(5,1,2) = [2,1,1,1,1,0], && T(5,2,1) = [10,5,3,2,1,0],\\
&T(5,3,0) = [30,15,7,3,1,0], && T(6,1,1) = [2,1,1,1,1,1],\\
&T(6,2,0) = [12,6,4,3,2,1].
\end{aligned}$$

[2]The authors wish to thank W. Zudilin for this information.

Therefore every Tornheim sum $T(m,k,n)$ has been expressed in terms of the set

$$\{T(i,0,N-i):\ 1 \leq i \leq N-2\}. \tag{6.28}$$

The value $T(1,0,N-1)$ is given in (6.2) and $T(N/2,0,N/2)$ appears in (6.4). Moreover, Euler's relation (6.3) reduces the number of unknown Tornheim sums to $N/2-2$. In the case $N=8$ the two unknowns are $T(6,0,2)$ and $T(5,0,3)$. Among the 15 sums discussed above, there are three with last index equal to 0, namely $T(6,2,0)$, $T(5,3,0)$ and $T(4,4,0)$. Each one of them produces an equation in the unknowns $T(6,0,2)$ and $T(5,0,3)$ coming from the evaluation

$$T(m,k,0) = \zeta(m)\zeta(k). \tag{6.29}$$

For instance, the case $T(4,4,0)$ gives

$$5T(6,0,2) + 2T(5,0,3) = \frac{163}{12}\zeta(8) - 8\zeta(3)\zeta(5). \tag{6.30}$$

This is the same relation among these sums obtained by Huard in (1.14). Unfortunately, the sums $T(6,2,0)$ and $T(5,3,0)$ yield the same relation, so we are unable to produce an analytic expression for all Tornheim sums of weight 8, free of an unevaluated integral.

We conclude that every Tornheim sum of weight 8 is a rational linear combination of the set

$$G_8 := \{\zeta(8),\ \zeta(3)\zeta(5),\ T(6,0,2)\}. \tag{6.31}$$

The table shows the corresponding coefficients:

$$
\begin{array}{ll}
T(1,0,7) = \left[\frac{5}{4}, -1, 0\right] & T(1,1,6) = \left[\frac{5}{2}, -2, 0\right] \\
T(2,0,6) = \left[\frac{2}{3}, 0, -1\right] & T(2,1,5) = \left[\frac{19}{6}, -2, -1\right] \\
T(2,2,4) = \left[\frac{19}{3}, -4, -2\right] & T(3,0,5) = \left[-\frac{187}{24}, 5, \frac{5}{2}\right] \\
T(3,1,4) = \left[-\frac{37}{8}, 3, \frac{3}{2}\right] & T(3,2,3) = \left[\frac{41}{24}, -1, -\frac{1}{2}\right] \\
T(3,3,2) = \left[\frac{41}{12}, -2, -1\right] & T(4,0,4) = \left[\frac{1}{12}, 0, 0\right] \\
T(4,1,3) = \left[-\frac{109}{24}, 3, \frac{3}{2}\right] & T(4,2,2) = \left[-\frac{17}{6}, 2, 1\right] \\
T(4,3,1) = \left[\frac{7}{12}, 0, 0\right] & T(4,4,0) = \left[\frac{7}{6}, 0, 0\right] \\
T(5,0,3) = \left[\frac{163}{24}, -4, -\frac{5}{2}\right] & T(5,1,2) = \left[\frac{9}{4}, -1, -1\right] \\
T(5,2,1) = \left[-\frac{7}{12}, 1, 0\right] & T(5,3,0) = [0, 1, 0] \\
T(6,0,2) = [0, 0, 1] & T(6,1,1) = \left[\frac{9}{4}, -1, 0\right] \\
T(6,2,0) = \left[\frac{5}{3}, 0, 0\right]. &
\end{array}
$$

The results derived in this paper give us the remaining unevaluated Tornheim sum $T(6,0,2)$ in terms of the integral $Y^*_{2,6}$ as

$$T(6,0,2) = \tfrac{7}{6}\zeta(8) - 6\zeta(3)\zeta(5) - Y^*_{2,6}. \tag{6.32}$$

Therefore, the generating set for Tornheim sums of weight 8 can also be taken as

$$G_8^* := \{\zeta(8),\, \zeta(3)\zeta(5),\, Y_{2,6}^*\}. \tag{6.33}$$

**Weight 10**

For weight 10, the algorithm follows step by step the previous case. We find that all Tornheim sums in $\mathscr{Z}_{10}^0$ are generated by the set

$$G_{10} := \{\zeta(10),\, \zeta^2(5),\, \zeta(3)\zeta(7), T(8,0,2)\}. \tag{6.34}$$

For example,

$$T(3,2,5) = -\tfrac{103}{40}\zeta(10) + \zeta^2(5) + \zeta(3)\zeta(7) + \tfrac{1}{2}T(8,0,2).$$

According to Theorem 2.3, the remaining unevaluated Tornheim sum $T(8,0,2)$ can be expressed in terms of the integral $Y_{2,8}^*$ as

$$T(8,0,2) = \tfrac{23}{20}\zeta(10) - 8\zeta(3)\zeta(7) - 4\zeta(5)^2 + Y_{2,8}^*. \tag{6.35}$$

In particular, we may also use the generating set

$$G_{10}^* := \{\zeta(10),\, \zeta^2(5),\, \zeta(3)\zeta(7), Y_{2,8}^*\}. \tag{6.36}$$

**Generating set for Tornheim sums of even weight**

The same algorithm described above can be used to produce a generating set for the Tornheim sums of even weight $N$. Except for $T(0,0,N) = \zeta(N-1) - \zeta(N)$, every such sum is a rational linear combination of the elements of the set

$$\left\{\zeta(N),\, \zeta(j)\zeta(N-j) :\, j \text{ odd }, 3 \leq j \leq 2\lfloor \frac{N-1}{4} \rfloor + 1 \right\} \tag{6.37}$$

and, for $N \geq 8$, one must also include the collection of integrals

$$\left\{ Y_{2r,N-2r}^* :\, 1 \leq r \leq \left\lfloor \frac{N-2}{6} \right\rfloor \right\}, \tag{6.38}$$

where where $Y_{2r,N-2r}^*$ is defined in (1.22).

The smallest weight for which one requires two basis Tornheim sums is 14. In this case the generating set is

$$G_{14} := \{\zeta(14),\, \zeta^2(7),\, \zeta(5)\zeta(9), \zeta(3)\zeta(11), T(12,0,2), T(10,0,4)\}. \tag{6.39}$$

The evaluation of $T(12,0,2)$ and $T(10,0,4)$ according to Theorem 2.3 gives

$$T(12,0,2) = \tfrac{481}{420}\zeta(14) - 12\zeta(3)\zeta(11) - 12\zeta(5)\zeta(9) - 6\zeta(7)^2 + Y_{2,12}^*,$$
$$T(10,0,4) = \tfrac{7}{12}\zeta(14) - 120\zeta(3)\zeta(11) - 60\zeta(5)\zeta(9) - 20\zeta(7)^2 + Y_{4,10}^*,$$

with

$$Y_{2,12}^* = Y_{2,12} + 12\zeta(13)\log 2\pi,$$
$$Y_{4,10}^* = Y_{4,10} + 220\zeta(13)\log 2\pi.$$

In particular, we may also use the generating set

$$G_{14}^* := \{\zeta(14),\, \zeta^2(7),\, \zeta(5)\zeta(9), \zeta(3)\zeta(11), Y_{2,12}^*, Y_{4,10}^*\}. \tag{6.40}$$

**Note 6.4.** In [1] the reader will find the sums

$$\sigma_h(s,t) := \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{1}{k^s} \frac{1}{n^t}, \tag{6.41}$$

that can be expressed as

$$\sigma_h(s,t) = \zeta(s)\zeta(t) - \zeta(s+t) - T(t,0,s). \tag{6.42}$$

The authors analyze a system of equations for the sums $\sigma_h(s,t)$ with the weight $w := s+t$ fixed. For $w$ odd, the system has full rank and they obtain Huard's expression for the Tornheim sums. In the case $w = 2n$ even, they establish that the dimension of the null space is $\lfloor (n-1)/3 \rfloor$, so every sum can be expressed in terms of this number of basis elements. In our case, $N = n/2$, thus the expected number of generators for all Tornheim sums of weight $N$ is at most $\lfloor (N-2)/6 \rfloor$. The fact that the set

$$\{ T(N-2r, 0, 2r) : 1 \leq r \leq \lfloor (N-2)/6 \rfloor \} \tag{6.43}$$

can be used to generate all Tornheim sums (aside from the usual product of zeta values) will follow from a careful analysis of the identities generated by the relations $T(m,n,0) = \zeta(m)\zeta(n)$. We leave the details for the ambitious reader. The fact that these sums are linearly independent is beyond our reach.

## 7. Conclusions

We have discussed an algorithm that evaluates all the Tornheim sums

$$T(m,k,n) := \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{r^m \, s^k \, (r+s)^n}, \tag{7.1}$$

of a given even weight $N := m+k+n$, as rational linear combinations of the value $\zeta(N)$, products of zeta values $\zeta(j)\zeta(N-j)$ with $j$ odd in the range $3 \leq j \leq 2\lfloor \frac{N-1}{4} \rfloor + 1$ and the integrals

$$\left\{ Y^{*}_{2r,N-2r} : 1 \leq r \leq \left\lfloor \frac{N-2}{6} \right\rfloor \right\}, \tag{7.2}$$

where

$$Y^{*}_{m,n} := \frac{2(2\pi)^{m+n-2}}{m!(n-2)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} X_{j,m+n-2-j} + (-1)^{N/2-1} \binom{N-2}{m-1} \zeta(N-1) \log 2\pi, \tag{7.3}$$

and the integral $X_{k,l}$ is defined by

$$X_{k,l} := (-1)^{\lfloor l/2 \rfloor} \frac{l!}{(2\pi)^l} \int_0^1 \log \Gamma(q) B_k(q) \, \mathrm{Cl}_{l+1}(2\pi q) \, dq. \tag{7.4}$$

Here $B_k$ is the Bernoulli polynomial and Cl is the Clausen function.

All the Tornheim sums of a given even weight $N$ can be expressed in terms of zeta values and a reduced number of basis sums of the type $T(N-2r, 0, 2r)$, with

$r = 1, \ldots, \lfloor \frac{N-2}{6} \rfloor$. These sums, in turn, can themselves be expressed in terms of zeta values and the integral $Y^*_{2r,N-2r}$, according to Theorem 2.3:

$$T(N-2r, 0, 2r) = (-1)^{N/2-1} Y^*_{2r,N-2r} + \zeta(2r)\zeta(N-2r) - \frac{1}{2}\zeta(N) - \sum_{j=1}^{N/2-2} \binom{N-2-2j}{2r-1} \zeta(2j+1)\zeta(N-1-2j). \tag{7.5}$$

Our results may perhaps be used to develop fast numerical codes to compute even weight Tornheim sums to high accuracy. Since the whole family of Tornheim sums of a given weight can be expressed in terms of zeta values and a small basis of Tornheim sums, it is enough to compute the basis sums to the required accuracy. This will involve the numerical calculation of the $Y$ integrals.

For example, of the 46 admissible triples at weight $N = 12$, 30 give Tornheim sums that depend on the value of the single $T(10, 0, 2)$. This sum is given by

$$T(10, 0, 2) = -Y^*_{2,10} - 10\zeta(5)\zeta(7) - 10\zeta(3)\zeta(9) + \tfrac{792}{691}\zeta(12). \tag{7.6}$$

A 30-digit precision Mathematica calculation of the integral $Y^*_{2,10}$ gives

$$T(10, 0, 2) = 0.645\,324\,784\,017\,496\,594\,071\,783\,081\,476\ldots. \tag{7.7}$$

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## References


[1] D. Borwein, J. M. Borwein, and R. Girgensohn. Explicit evaluation of Euler sums. *Proc. Edin. Math. Soc.*, 38:277–294, 1995.

[2] J. M. Borwein, D. H. Bailey, and R. Girgensohn. *Experimentation in Mathematics: Computational Paths to Discovery.* A. K. Peters, 1st edition, 2004.

[3] J. M. Borwein and D. Bradley. Thirty-two Goldbach variations. *International Journal of Number Theory*, 2:65–103, 2006.

[4] O. Espinosa and V. Moll. On some definite integrals involving the Hurwitz zeta function. Part 1. *The Ramanujan Journal*, 6:159–188, 2002.

[5] O. Espinosa and V. Moll. On some definite integrals involving the Hurwitz zeta function. Part 2. *The Ramanujan Journal*, 6:449–468, 2002.

[6] O. Espinosa and V. Moll. A generalized polygamma function. *Integral Transforms and Special Functions*, 15:101–115, 2004.

[7] O. Espinosa and V. Moll. The evaluation of Tornheim double sums. Part 1. *Journal of Number Theory*, 116:200–229, 2006.

[8] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

[9] A. Granville. A decomposition of Riemann's zeta function. *London Mathematical Society*, 247:95–101, 1997.

[10] J. Huard, K. Williams, and N. Zhang. On Tornheim's double series. *Acta Arith.*, 75:105–117, 1996.

[11] L. Tornheim. Harmonic double series. *Amer. J. Math.*, pages 303–314, 1950.

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