AN ELEMENTARY TRIGONOMETRIC EQUATION

VICTOR H. MOLL

ABSTRACT. We provide a systematic study of the trigonometric equation

$$A \tan a + B \sin b = C.$$

In particular, the classical example

$$\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \sqrt{11},$$

appears in a natural form $\sqrt{11}$. A second proof involving Gaussian sums is also discussed.

1. Introduction

In the Problems and Solutions section of the September 2006 issue of The College Mathematics Journal [7], one is told of the identity

$$\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \sqrt{11}, \tag{1.1}$$

that appeared as Problem 218 of the same journal [1]. The problem asks to prove the related identites

$$\tan \frac{\pi}{11} + 4 \sin \frac{3\pi}{11} = \sqrt{11}, \tag{1.2}$$

and

$$\tan \frac{4\pi}{11} + 4 \sin \frac{\pi}{11} = \sqrt{11}. \tag{1.3}$$

These evaluations have appeared in classical texts. For instance, (1.1) is an exercise in Hobson's Trigonometry text [6] (pages 123 and 382). It also appears as Exercise 14 on page 270 of T. J. I. Bromwich's treatise on Infinite Series [4].

During the period 1991-2003 it was my privilege to work with George Boros. First as an advisor and then as colleague. Our method of work was rather unorthodox. George has a very good knowledge of numbers, so he will bring me many pages of formulas and my role was to try to figure out where these things fit. Around 1997 he showed me the '$\sqrt{11}$ problem'. This note is a reflection of what I learned from him.

2. The reduction

We consider the equation

$$A \tan a + B \sin b = C, \tag{2.1}$$

Date: January 15, 2007.

Key words and phrases. Quadratic radicals, Gauss sums.

where A, B and C^2 are *rational* numbers. We may assume that $A = 1$ and we write (2.1) as

$$\sin a + B\cos a \sin b = C\cos a. \tag{2.2}$$

The procedure described here will give *some* solutions of (2.2). In particular, the radical $\sqrt{11}$ will appear in a natural form.

The identity

$$\cos a \sin b = \tfrac{1}{2}\left(\sin(a+b) - \sin(a-b)\right), \tag{2.3}$$

converts (2.2) into

$$\sin a + \tfrac{1}{2}B\sin(a+b) - \tfrac{1}{2}B\sin(a-b) = C\cos a. \tag{2.4}$$

Lemma 2.1. Assume the angles a, b, c satisfy (2.4). Then

$$\begin{aligned} \tfrac{1}{4}(2-2C^2+B^2) &= \tfrac{1}{4}(2+2C^2-B^2)\cos(2a) + \tfrac{1}{8}B^2\cos(2a+2b) + \\ &+ \tfrac{1}{8}B^2\cos(2a-2b) + \tfrac{1}{2}B\cos(2a+b) + \tfrac{1}{4}B^2\cos(2b) \\ &- \tfrac{1}{2}B\cos(2a-b). \end{aligned}$$

Proof. Square (2.4) and use

$$\begin{aligned} \sin u \sin v &= \tfrac{1}{2}\cos(u-v) - \tfrac{1}{2}\cos(u+v), \\ \cos u \cos v &= \tfrac{1}{2}\cos(u-v) + \tfrac{1}{2}\cos(u+v), \end{aligned}$$

the standard formulas for double angle

$$\sin^2 u = \tfrac{1}{2}(1-\cos(2a)) \text{ and } \cos^2 u = \tfrac{1}{2}(1+\cos(2a)), \tag{2.5}$$

and

$$\sin^2(a+b) + \sin^2(a-b) = 1 - \cos(2a)\cos(2b) \tag{2.6}$$

to produce the result. □

3. Some special cases

In order to produce some special solutions of the equation in Lemma 2.1, we begin by matching the different coefficients appearing in it. For example, forcing

$$\tfrac{1}{8}B^2 = \tfrac{1}{2}B, \tag{3.1}$$

that is, choosing $B = 4$, converts it into

$$\begin{aligned} \tfrac{1}{2}(9-C^2) &= \tfrac{1}{2}(C^2-7)\cos(2a) + 2\cos(2a+2b) + 2\cos(2a-2b) + \\ &\quad +2\cos(2a+b) + 4\cos(2b) - 2\cos(2a-b). \end{aligned} \tag{3.2}$$

We now simplify this equation further by imposing the condition

$$\tfrac{1}{2}(C^2-7) = 2, \tag{3.3}$$

to make almost every coefficient on the right hand side of (3.2) equal to ± 2. This produces $C = \pm\sqrt{11}$.

The equation (3.2) becomes

$$\begin{aligned} -\tfrac{1}{2} &= \cos(2a-2b) - \cos(2a-b) + \cos(2a) + \cos(2a+b) + \\ &+ \cos(2a+2b) + 2\cos(2b). \end{aligned} \tag{3.4}$$

Observe that five out the six angles appearing in (3.4) are in arithmetic progression: $2a-2b$, $2a-b$, $2a$, $2a+b$ and $2a+2b$. We now proceed to establish further restrictions on the angles a and b so that the final angle, namely $2b$, is also part of this progression. This will determine *some* solutions of the original problem.

3.1. **The first example.** We assume first $2b = 2a - b$, that is, $2a = 3b$. Then (3.4) becomes

$$\cos(2a-2b)+\cos(2a-b)+\cos(2a)+\cos(2a+b)+\cos(2a+2b) = -\tfrac{1}{2}. \tag{3.5}$$

This identity is simplified using the closed form expression for sums of cosines in arithmetic progression:

$$\sum_{k=0}^{n-1} \cos(x+ky) = \frac{\cos(x+(n-1)y/2)\,\sin(ny/2)}{\sin(y/2)}. \tag{3.6}$$

This formula is easy to establish and it can be found as 1.341.3 in the table [5]. In our first case, we have $x = 2a-2b, y = b$ and $n = 5$ so that (3.6) produces

$$\frac{\sin(5b/2)\,\cos(2a)}{\sin(b/2)} = -\tfrac{1}{2}. \tag{3.7}$$

We exclude the case where b is an integer multiple of 2π, since in that situation equation (2.1) is not interesting. Using $2a = 3b$ and the elementary identity

$$\sin x \cos y = \tfrac{1}{2}\left(\sin(x+y)+\sin(x-y)\right), \tag{3.8}$$

(3.7) produces

$$\sin\left(\frac{11b}{2}\right) = 0 \text{ and } \sin\left(\frac{b}{2}\right) \neq 0. \tag{3.9}$$

We conclude that

$$b = \frac{2k\pi}{11},\ k \in \mathbb{Z},\ k \not\equiv 0 \bmod 11, \tag{3.10}$$

and the angle a is given by

$$a = \frac{3k\pi}{11},\ k \in \mathbb{Z},\ k \not\equiv 0 \bmod 11. \tag{3.11}$$

Therefore, for $k \in \mathbb{Z}$ not divisible by 11, we have found some solutions to (2.1):

$$\tan\left(\frac{3k\pi}{11}\right) + 4\sin\left(\frac{2k\pi}{11}\right) = \pm\sqrt{11}. \tag{3.12}$$

A numerical computation of the left hand side shows that, modulo 11, $k = 1, 3, 4, 5, 9$ correspond to the positive sign and $k = 2, 6, 7, 8, 10$ to the minus sign. Reducing the angle to the smallest integer multiple of $\pi/11$ we obtain the five relations described in the next theorem, that include (1.1), (1.2), and (1.3).

Theorem 3.1. *The following identities hold*

$$\begin{aligned} \tan\left(\frac{\pi}{11}\right) + 4\sin\left(\frac{3\pi}{11}\right) &= \sqrt{11}, \\ \tan\left(\frac{2\pi}{11}\right) - 4\sin\left(\frac{5\pi}{11}\right) &= -\sqrt{11}, \\ \tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right) &= \sqrt{11}, \\ \tan\left(\frac{4\pi}{11}\right) + 4\sin\left(\frac{\pi}{11}\right) &= \sqrt{11}. \\ \tan\left(\frac{5\pi}{11}\right) - 4\sin\left(\frac{4\pi}{11}\right) &= \sqrt{11}. \end{aligned} \tag{3.13}$$

3.2. **A second example.** We now assume $a = b$. Then (3.4) becomes

$$\tfrac{1}{4}(5 - C^2) = -\cos a + \tfrac{1}{4}(C^2+1) + \cos(2a) + \cos(3a) + \cos(4a). \tag{3.14}$$

Choose $C^2 = 3$ to make all the coefficients on the right hand side ± 1. This yields

$$-\cos a + \cos(2a) + \cos(3a) + \cos(4a) = \tfrac{1}{2}, \tag{3.15}$$

that we write as

$$1 + \cos a + \cos(2a) + \cos(3a) + \cos(4a) = \tfrac{1}{2}(3 + 4\cos a). \tag{3.16}$$

Using (3.6) we obtain

$$\frac{\sin(5a/2)\,\cos(2a)}{\sin(a/2)} = \tfrac{1}{2}(3 + 4\cos a), \tag{3.17}$$

that is equivalent to

$$\sin(9a/2) = 2\sin(a/2)\left[3 - 4\sin^2(a/2)\right]. \tag{3.18}$$

Now use $\sin(3t) = 3\sin t - 4\sin^3 t$ to write (3.18) as

$$\sin(9a/2) = 2\sin(3a/2), \quad \text{ and } \sin(a/2) \neq 0. \tag{3.19}$$

Let $x = 3a/2$ to write

$$\sin x\,(4\sin^2 x - 1) = 0, \tag{3.20}$$

that has solutions $x = \frac{\pi}{6} \times \{1,\, 5,\, 7,\, 11\} + 2m\pi$. Therefore

$$a = \frac{\pi}{9} \times \{1,\, 5,\, 6,\, 7,\, 11,\, 12\} + \frac{12m\pi}{9}. \tag{3.21}$$

This yields a second family of solutions to (2.1).

Theorem 3.2. *The following identities hold*

$$\begin{aligned} \tan\left(\frac{\pi}{9}\right) + 4\sin\left(\frac{\pi}{9}\right) &= \sqrt{3}, \\ \tan\left(\frac{2\pi}{9}\right) - 4\sin\left(\frac{2\pi}{9}\right) &= -\sqrt{3}, \\ \tan\left(\frac{4\pi}{9}\right) - 4\sin\left(\frac{4\pi}{9}\right) &= \sqrt{3}, \\ \tan\left(\frac{6\pi}{9}\right) + 4\sin\left(\frac{6\pi}{9}\right) &= \sqrt{3}. \end{aligned} \tag{3.22}$$

Using the same techniques the reader is invited to check the following result.

Theorem 3.3. *The following identities hold*

$$\tan\left(\frac{\pi}{7}\right) - 4\sin\left(\frac{2\pi}{7}\right) = -\sqrt{7}, \tag{3.23}$$

$$\tan\left(\frac{2\pi}{7}\right) - 4\sin\left(\frac{3\pi}{7}\right) = -\sqrt{7},$$

$$\tan\left(\frac{3\pi}{7}\right) - 4\sin\left(\frac{\pi}{7}\right) = \sqrt{7}.$$

4. A proof using Gaussian sums

In this section we present a proof of identity (1.1) using the value of the Gaussian sum

$$G_n = \sum_{j=0}^{n-1} e^{2\pi i j^2/n}. \tag{4.1}$$

Gauss proved that

$$G_n = \begin{cases} (1+i)\sqrt{n} & \text{if } n \equiv 0 \bmod 4, \\ \sqrt{n} & \text{if } n \equiv 1 \bmod 4, \\ 0 & n \text{ if } \equiv 2 \bmod 4, \\ i\sqrt{n} & \text{if } \quad n \equiv 3 \bmod 4. \end{cases} \tag{4.2}$$

In particular, for $n = 11$, we have

$$G_{11} = i\sqrt{11}. \tag{4.3}$$

The reader will find a proof of (4.2) in [2] and much more information about these sums in [3].

Let $x = e^{2\pi i/11}$, so that $x^{11} = 1$. This yields

$$1 + x + x^2 + \cdots + x^9 + x^{10} = 0. \tag{4.4}$$

Gauss' result gives

$$1 + 2(x + x^3 + x^4 + x^5 + x^9) = i\sqrt{11}, \tag{4.5}$$

where the numbers 1, 3, 4, 5, 9 are the squares modulo 11, also called *quadratic residues*.

Using $\sin t = (e^{it} - e^{-it})/2i$ and $\cos t = (e^{it} + e^{-it})/2$ we obtain

$$4i \sin\frac{2\pi}{11} = 2(x - x^{-1}) = 2(x - x^{10}), \tag{4.6}$$

and

$$i\tan\frac{3\pi}{11} = \frac{x^{3/2}-x^{-3/2}}{x^{3/2}+x^{-3/2}} \tag{4.7}$$
$$= \frac{x^3-1}{x^3+1}$$
$$= \frac{x^3-x^{33}}{x^3+1}$$
$$= x^3(1-x^{15})\frac{1+x^{15}}{1+x^3}$$
$$= x^3(1-x^4)\left[1-x^3+x^6-x^9+x^{12}\right]$$
$$= x^3(1-x^4)\left[1+x-x^3+x^6-x^9\right]$$
$$= -x-x^2+x^3+x^4+x^5-x^6-x^7-x^8+x^9+x^{10}.$$

Therefore

$$i\tan\frac{3\pi}{11}+4i\sin\frac{2\pi}{11} = x+x^3+x^4+x^5+x^9-(x^2+x^6+x^7+x^8+x^{10})$$
$$= 1+2(x+x^3+x^4+x^5+x^9)-(1+x+x^2+\cdots+x^{10})$$
$$= 1+2(x+x^3+x^4+x^5+x^9)$$
$$= i\sqrt{11}.$$

The proof of (1.1) is complete.

This technique can be used to establish (1.2), (1.3) and many other identities. The reader is invited to check

$$\tan\frac{2\pi}{7}+4\sin\frac{2\pi}{7}-4\sin\frac{\pi}{7}=\sqrt{7}, \tag{4.8}$$

and

$$\tan\frac{4\pi}{19}+4\sin\frac{5\pi}{19}-4\sin\frac{6\pi}{19}+4\sin\frac{9\pi}{19}=\sqrt{19}, \tag{4.9}$$

and also

$$\tan\frac{\pi}{9}+2\sin\frac{\pi}{9}-2\sin\frac{2\pi}{9}+2\sin\frac{4\pi}{9}=\sqrt{3}, \tag{4.10}$$

by these methods.

Acknowledgments. The author wishes to thank T. Amdeberham for a careful reading of the manuscript. The partial support of NSF-DMS 0409968 is acknowledged.

References

[1] Problem 218. *The College Mathematics Journal*, 14:358, 1983.
[2] T. Apostol. *Introduction to Analytic Number Theory*. Springer-Verlag, New York, 1976.
[3] B. B. Berndt, R. Evans, and K. S. Williams. *Gauss and Jacobi sums*. Canadian Mathematical Society, 1998.
[4] T. J. Bromwich. *An introduction to the theory of infinite series*. Chelsea Publishing Company, New York, N. Y., 3rd edition, 1991.
[5] I. S. Gradshteyn and I. M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

[6] E. W. Hobson. *A treatise on plane and advanced Trigonometry*. Dover Publications, Inc., New York, 7th edition, 1928.

[7] S. Rabinowitz. Problem 831. *The College Mathematics Journal*, 37(4):831, 2006.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu