RAMANUJAN'S MASTER THEOREM

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Abstract. S. Ramanujan introduced a technique, known as Ramanujan's Master Theorem, which provides an analytic expression for the Mellin transform of a function. The main identity of this theorem involves the extrapolation of the sequence of coefficients of the integrand, defined originally as a function on $\mathbb{N}$ to $\mathbb{C}$. The history and proof of this result are reviewed. Applications to the evaluation of a variety of definite integrals is presented.

1. Introduction

Ramanujan's Master Theorem refers to the formal identity

$$\int_0^\infty x^{s-1} \left\{ \lambda(0) - \frac{x}{1!}\lambda(1) + \frac{x^2}{2!}\lambda(2) - \cdots \right\} dx = \Gamma(s)\lambda(-s) \tag{1.1}$$

stated by S. Ramanujan's in his *Quarterly Reports* [3, p. 298]. It was widely used by him as a tool in computing definite integrals and infinite series. In fact, as G. H. Hardy puts it in [16], he "was particularly fond of them [(1.1) and (2.4)], and used them as one of his commonest tools."

The goal of this semi-expository paper is to discuss the history of (1.1) and to describe a selection of definite integrals that can be evaluated by this technique. Section 2 discusses evidence that (1.1) was nearly discovered as early as 1847 by J. W. L. Glaisher and J. O'Kinealy. Section 3 briefly outlines Hardy's proof of Ramanujan's Master Theorem. The critical issue is the extension of the function λ from $\mathbb{N}$ to $\mathbb{C}$. Section 4 presents the evaluation of a collection of definite integrals with most of the examples coming from the classical table [15]. Section 5 is a recollect on the evaluation of the quartic integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{1.2}$$

This section provides a personal historical context: it was the evaluation of (1.2) that lead one of the authors to (1.1). Section 6 contains integrals derived from classical polynomials. The last three sections outline the use

Date: February 22, 2011.

2000 *Mathematics Subject Classification.* Primary 33B15, Secondary 33C05, 33C45.

Key words and phrases. Integrals, analytic continuation, series representation, hypergeometric functions, random walk integrals, method of brackets.

of Ramanujan's Master Theorem to ongoing research projects: Section 7 deals with an integral related to the distance traveled by a uniform random walk in a fixed number of steps; Section 8 describes an application to obtain an analytic extensions of a sequence and finally Section 9 presents a multi-dimensional version of the main theorem that has appeared in the context of Feynman diagrams.

The use of Ramanujan's Master Theorem has been restricted here mostly to the evaluation of definite integrals. Many other applications appear in the literature. For instance, Ramanujan himself employed it to derive various expansions: the two examples given in [16, 11.9] are the expansion of e^{-ax} in powers of xe^{bx} as well as an expansion of the powers x^r of a root of $aqx^p + x^q = 1$ in terms of powers of a.

2. History

The first integral theorem in the spirit of Ramanujan's Master Theorem appears to have been given by Glaisher in 1874, [12]:

$$\int_0^\infty \left(a_0 - a_1 x^2 + a_2 x^4 - \dots \right) \, dx = \frac{\pi}{2} a_{-\frac{1}{2}}. \tag{2.1}$$

Glaisher writes, "of course, a_n being only defined for n a positive integer, $a_{-\frac{1}{2}}$ is without meaning. But in cases where a_n involves factorials, there is a strong presumption, derived from experience in similar questions, that the formula will give correct results if the continuity of the terms is preserved by the substitution of gamma functions for the factorials. This I have found to be true in every case to which I have applied (2.1)."

Glaisher in [12] formally obtained (2.1) by integrating term-by-term the identity

$$a_0 - a_1 x^2 + a_2 x^4 - \cdots = \frac{a_0}{1+x^2} - \Delta a_0 \frac{x^2}{(1+x^2)^2} + \Delta^2 a_0 \frac{x^4}{(1+x^2)^3} - \cdots . \tag{2.2}$$

Here Δ is the forward-difference operator defined by $\Delta a_n = a_{n+1} - a_n$.

Glaisher's argument, published in July 1874, was picked up in October of the same year by O'Kinealy who critically simplified it in [17]. Employing the forward-shift operator E defined by $E \cdot \lambda(n) = \lambda(n+1)$, O'Kinealy writes the left-hand side of (2.2) as $\frac{1}{1+x^2 E} \cdot a_0$ which he then integrates treating E as a number to obtain

$$\frac{\pi}{2} E^{-1/2} \cdot a_0 = \frac{\pi}{2} a_{-\frac{1}{2}},$$

thus arriving at the identity (2.1). O'Kinealy, [17], remarks that "it is evident that there are numerous theorems of the same kind". As an example, he proposes integrating $\cos(xE) \cdot a_0$ and $\sin(xE) \cdot a_0$.

O'Kinealy's improvements are emphatically received by Glaisher in a short letter [11] to the editors in which he remarks that he had examined O'Kinealy's work and that, "after developing the method so far as to include these formulae and several others, I communicated it, with the examples, to Professor Cayley, in a letter on the 22nd or 23rd of July, which gave rise to a short correspondence between us on the matter at the end of July. My only reason for wishing to mention this at once is that otherwise, as I hope soon to be able to return to the subject and somewhat develop the principle, which is to a certain extent novel, it might be thought at some future time that I had availed myself of Mr. O'Kinealy's idea without proper acknowledgement."

Unfortunately, no further work seems to have appeared along these lines so that one can only speculate as to what Glaisher and Cayley have figured out. It is not unreasonable to guess that they might very well have developed an idea somewhat similar to Ramanujan's Master Theorem (1.1). In fact, just slightly generalizing O'Kinealy's argument is enough to formally obtain (1.1). This is shown next.

Formal proof of (1.1).

$$\begin{aligned}\int_0^\infty x^{s-1}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\lambda(n)x^n\,dx &= \int_0^\infty x^{s-1}\sum_{n=0}^\infty \frac{(-1)^n}{n!}E^n x^n\,dx\cdot\lambda(0)\\ &= \int_0^\infty x^{s-1}e^{-Ex}\,dx\cdot\lambda(0)\\ &= \frac{\Gamma(s)}{E^s}\cdot\lambda(0)\\ &= \Gamma(s)\lambda(-s)\end{aligned}$$

where in the penultimate step we employed the integral representation

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx \tag{2.3}$$

of the gamma function and treated the operator E as a number. It is this step which renders the proof formal: clearly the coefficient function $\lambda(n)$ needs to satisfy certain conditions for the result to be valid. This will be discussed in Section 3. □

The identity

$$\int_0^\infty x^{s-1}\left\{\varphi(0) - x\varphi(1) + x^2\varphi(2) - \cdots\right\}\,dx = \frac{\pi}{\sin s\pi}\varphi(-s), \tag{2.4}$$

is given by Ramanujan alongside (1.1) (see [3]). The formulations are equivalent: the relation $\varphi(n) = \lambda(n)/\Gamma(n+1)$ converts (2.4) into (1.1).

The integral theorem (2.1) also appears in the text [9] as Exercise 7 on Chapter XXVI. It is attributed there to Glaisher. The exercise asks to show (2.1) and to "apply this theorem to find $\int_0^\infty \frac{\sin ax}{x}\,dx$."

The argument that Ramanujan gives for (1.1) appears in Hardy [16] where the author demonstrates that, while the argument can be made rigorous in certain cases, it usually leads to false intermediate formulae which "excludes practically all of Ramanujan's examples".

A rigorous proof of (1.1) and its special case (2.1) was given in Chapter XI of [16]. This text is based on a series of lectures on Ramanujan's work given in the Fall semester of 1936 at Harvard University.

3. Rigorous treatment of the Master Theorem

The proof of Ramanujan's Master Theorem provided by Hardy in [16] employs Cauchy's residue theorem as well as the well-known Mellin inversion formula which is recalled next followed by an outline of the proof.

Theorem 3.1 (Mellin inversion formula)**.** *Assume that $F(s)$ is analytic in the strip $a < \text{Re } s < b$ and define f by*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} F(s) x^{-s} \, ds.$$

If this integral converges absolutely and uniformly for $c \in (a, b)$ then

$$F(s) = \int_0^\infty x^{s-1} f(x) \, dx.$$

Theorem 3.2 (Ramanujan's Master Theorem)**.** *Let $\varphi(z)$ be an analytic (single-valued) function, defined on a half-plane*

$$H(\delta) = \{z \in \mathbb{C} : \text{Re } z \geq -\delta\} \tag{3.1}$$

for some $0 < \delta < 1$. Suppose that, for some $A < \pi$, φ satisfies the growth condition

$$|\varphi(v + iw)| < Ce^{Pv + A|w|} \tag{3.2}$$

for all $z = v + iw \in H(\delta)$. Then (2.4) holds for all $0 < \text{Re } s < \delta$, that is

$$\int_0^\infty x^{s-1} \left\{ \varphi(0) - x\varphi(1) + x^2\varphi(2) - \cdots \right\} dx = \frac{\pi}{\sin s\pi} \varphi(-s). \tag{3.3}$$

Proof. Let $0 < x < e^{-P}$. The growth conditions show that the series

$$\Phi(x) = \varphi(0) - x\varphi(1) + x^2\varphi(2) - \cdots$$

converges. The residue theorem yields

$$\Phi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} \frac{\pi}{\sin \pi s} \varphi(-s) x^{-s} \, ds \tag{3.4}$$

for any $0 < c < \delta$. Observe that $\pi/\sin(\pi s)$ has poles at $s = -n$ for $n = 0, 1, 2, \ldots$ with residue $(-1)^n$. The integral in (3.4) converges absolutely and uniformly for $c \in (a, b)$ for any $0 < a < b < \delta$. The claim now follows from Theorem 3.1. □

Remark 3.3. The conversion $\varphi(u) = \lambda(u)/\Gamma(u + 1)$ establishes Ramanujan's Master Theorem in the form (1.1). The condition $\delta < 1$ ensures convergence of the integral in (3.3). Analytic continuation may be employed to validate (3.3) to a larger strip in which the integral converges. See also Section 8.

4. A collection of elementary examples

This section contains a collection of definite integrals that can be evaluated directly from Ramanujan's Master Theorem 3.2. For the convenience of the reader, the main theorem in the form (1.1) is reproduced below. Its hypothesis are described in Section 3.

Theorem 4.1. *Assume f admits an expansion of the form*

$$f(x) = \sum_{k=0}^{\infty} \frac{\lambda(k)}{k!} (-x)^k. \tag{4.1}$$

Then, the Mellin transform of f is given by

$$\int_0^\infty x^{s-1} f(x)\, dx = \Gamma(s)\lambda(-s). \tag{4.2}$$

Example 4.2. Let $a > 0$ and $f(x) = e^{-ax}$. Then

$$f(x) = \sum_{k=0}^{\infty} a^k \frac{(-x)^k}{k!} \tag{4.3}$$

so that $\lambda(k) = a^k$. Theorem 4.1 yields

$$\int_0^\infty x^{s-1} e^{-ax}\, dx = \frac{\Gamma(s)}{a^s}. \tag{4.4}$$

This evaluation simply shows the consistency of Ramanujan's Master Theorem with a scaled version of (2.3).

Example 4.3. Instances of series expansions involving factorials are particularly well-suited for the application of Ramanujan's Master Theorem. This is a consecuence of the analytic extension of factorials by the gamma function. To illustrate this fact, use the binomial theorem for $a > 0$ in the form

$$(1+x)^{-a} = \sum_{k=0}^{\infty} \binom{k+a-1}{k} (-x)^k = \sum_{k=0}^{\infty} \frac{\Gamma(k+a)}{\Gamma(a)} \frac{(-x)^k}{k!}. \tag{4.5}$$

Ramanujan's Master Theorem (1.1), with $\lambda(k) = \Gamma(a+k)/\Gamma(a)$, then yields

$$\int_0^\infty \frac{x^{s-1}\, dx}{(1+x)^a} = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}. \tag{4.6}$$

The right-hand side is recognized as $B(s, a-s)$, where B is the beta integral.

Example 4.4. Many of the functions appearing in this paper are special cases of the hypergeometric function

$${}_pF_q(\mathbf{c}; \mathbf{d}; -x) = \sum_{k=0}^{\infty} \frac{(c_1)_k\, (c_2)_k \cdots (c_p)_k}{(d_1)_k\, (d_2)_k \cdots (d_q)_k} \frac{(-x)^k}{k!}. \tag{4.7}$$

To apply Ramanujan's Master Theorem, write the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$ as $(a)_k = \Gamma(a+k)/\Gamma(a)$. It follows that

$$\int_0^\infty x^{s-1} {}_pF_q(\mathbf{c};\mathbf{d};-x)\,dx = \Gamma(s)\frac{\Gamma(c_1-s)\,\cdots\Gamma(c_p-s)\Gamma(d_1)\cdots\Gamma(d_q)}{\Gamma(c_1)\,\cdots\Gamma(c_p)\Gamma(d_1-s)\cdots\Gamma(d_q-s)}. \tag{4.8}$$

This appears as entry 7.511 in [15].

Example 4.5. The error function

$$\operatorname{erf} x := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \tag{4.9}$$

admits the hypergeometric representation

$$\operatorname{erf} x = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\tfrac{1}{2};\,\tfrac{3}{2};\,-x^2\right). \tag{4.10}$$

Therefore, its Mellin transform

$$\begin{aligned}\int_0^\infty x^{s-1}\operatorname{erf} x\,dx &= \frac{2}{\sqrt{\pi}}\int_0^\infty x^s\,{}_1F_1\left(\tfrac{1}{2};\,\tfrac{3}{2};\,-x^2\right)\,dx\\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{(s+1/2)-1}\,{}_1F_1\left(\tfrac{1}{2};\,\tfrac{3}{2};\,-t\right)\,dt\end{aligned}$$

is obtained from (4.8) as

$$\int_0^\infty x^{s-1}\operatorname{erf} x\,dx = -\frac{1}{s\sqrt{\pi}}\Gamma\left(\frac{1+s}{2}\right). \tag{4.11}$$

The reader will find in [1] a selection of definite integrals involving this error function.

Example 4.6. The Bessel function $J_\nu(x)$ admits the hypergeometric representation

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\frac{x^\nu}{2^\nu}\,{}_0F_1\left(-;\nu+1;-\frac{x^2}{4}\right). \tag{4.12}$$

Its Mellin transform

$$\begin{aligned}\int_0^\infty x^{s-1}J_\nu(x)\,dx &= \frac{1}{2^\nu\,\Gamma(\nu+1)}\int_0^\infty x^{s+\nu-1}{}_0F_1\left(-;\nu+1;-\frac{x^2}{4}\right)\,dx\\ &= \frac{2^{s-1}}{\Gamma(\nu+1)}\int_0^\infty t^{(s+\nu)/2-1}{}_0F_1\left(-;\nu+1;-t\right)\,dt\end{aligned}$$

is obtained from (4.8) as

$$\int_0^\infty x^{s-1}J_\nu(x)\,dx = \frac{2^{s-1}\Gamma\left(\frac{s+\nu}{2}\right)}{\Gamma\left(\frac{\nu-s}{2}+1\right)}. \tag{4.13}$$

This formula appears as 6.561.14 in [15].

Example 4.7. The expansion

$$\frac{\cos(t\tan^{-1}\sqrt{x})}{(1+x)^{t/2}} = \sum_{k=0}^{\infty} \frac{\Gamma(t+2k)\,\Gamma(k+1)}{\Gamma(t)\,\Gamma(2k+1)} \frac{(-x)^k}{k!}.$$

was established in [4] in the process of evaluating of a class of definite integrals. A direct application of Ramanujan's Master Theorem now yields

$$\int_0^\infty x^{\nu-1} \frac{\cos(2t\tan^{-1}\sqrt{x})}{(1+x)^t}\,dx = \frac{\Gamma(2t-2\nu)\,\Gamma(1-\nu)\,\Gamma(\nu)}{\Gamma(2t)\,\Gamma(1-2\nu)},$$

and $x=\tan^2\theta$ gives

$$\int_0^{\pi/2} \sin^\mu\theta\cos^{2t-\mu}\theta\cos(2t\theta)\,d\theta = \frac{\pi\Gamma(2t-\mu-1)}{2\sin(\pi\mu/2)\,\Gamma(2t)\Gamma(-\mu)}. \tag{4.14}$$

Similarly, the expansion

$$\frac{\sin(2t\tan^{-1}\sqrt{x})}{\sqrt{x}(1+x)^t} = \sum_{k=0}^{\infty} \frac{\Gamma(2t+2k+1)\,\Gamma(k+1)}{\Gamma(2t)\Gamma(2k+2)} \frac{(-x)^k}{k!}$$

produces

$$\int_0^{\pi/2} \sin^{\mu-1}\theta\cos^{2t-\mu}\theta\,\sin(2t\theta)\,d\theta = \frac{\pi\Gamma(2t-\mu)}{2\sin(\pi\mu/2)\Gamma(2t)\Gamma(1-\mu)}. \tag{4.15}$$

Example 4.8. The Mellin transform of the function $\log(1+x)/(1+x)$ is obtained from the expansion

$$\frac{\log(1+x)}{1+x} = -\sum_{k=1}^{\infty} H_k(-x)^k, \tag{4.16}$$

where $H_k = 1+\frac{1}{2}+\cdots+\frac{1}{k}$ is the harmonic number. In order to apply Ramanujan's Master Theorem it is required to extend H_k from $k\in\mathbb{N}$ to $k\in\mathbb{C}$. This is achieved by the relation

$$H_k = \gamma+\psi(k+1), \tag{4.17}$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function and $\gamma=-\Gamma'(1)$ is the Euler constant. The identity

$$\sum_{k=1}^{\infty} k!\psi(k+1)\,\frac{(-x)^k}{k!} = \frac{1}{1+x}\left(\gamma x-\log(1+x)\right) \tag{4.18}$$

now gives

$$\int_0^\infty \frac{x^{\nu-1}}{1+x}\log(1+x)\,dx = -\frac{\pi}{\sin\pi\nu}\left(\gamma+\psi(1-\nu)\right). \tag{4.19}$$

The special case $\nu=\frac{1}{2}$ yields

$$\int_0^\infty \frac{(\gamma+\log(1+x))}{\sqrt{x}\,(1+x)}\,dx = \pi(\gamma+2\log 2),$$

that produces the logarithmic integral

$$\int_0^\infty \frac{\log(1+t^2)}{1+t^2}\,dt = \pi \log 2. \tag{4.20}$$

This is equivalent to the classic evaluation

$$\int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2}\log 2, \tag{4.21}$$

given by Euler.

Example 4.9. The infinite product representation of the gamma function

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{n=1}^\infty \left(1+\frac{x}{n}\right)^{-1} e^{x/n} \tag{4.22}$$

produces the expansion

$$\log\Gamma(1+x) = -\gamma x + \sum_{k=2}^\infty \frac{\zeta(k)}{k}(-x)^k. \tag{4.23}$$

This may be written as

$$\frac{\log\Gamma(1+x)+\gamma x}{x^2} = \sum_{k=0}^\infty \lambda(k)\frac{(-x)^k}{k!} \tag{4.24}$$

with

$$\lambda(k) = \frac{\Gamma(k+1)\zeta(k+2)}{k+2}. \tag{4.25}$$

Ramanujan's Master Theorem implies

$$\int_0^\infty x^{\nu-1}\frac{\log\Gamma(1+x)+\gamma x}{x^2}\,dx = \frac{\pi}{\sin \pi\nu}\frac{\zeta(2-\nu)}{2-\nu}, \tag{4.26}$$

valid for $0<\nu<1$. The special cases $\nu = \frac{1}{2}$ and $\nu = \frac{3}{4}$ give

$$\int_0^\infty \frac{\log\Gamma(1+x)+\gamma x}{x^{5/2}}\,dx = \frac{2\pi}{3}\zeta(3/2) \tag{4.27}$$

and

$$\int_0^\infty \frac{\log\Gamma(1+x)+\gamma x}{x^{5/4}}\,dx = \frac{4}{5}\sqrt{2}\,\pi\,\zeta(5/4), \tag{4.28}$$

respectively. `Mathematica` 7 is unable to evaluate these examples.

Example 4.10. Differentiating (4.23) yields

$$\psi(x+1) = -\gamma + \sum_{k=1}^\infty (-1)^{k+1}\zeta(k+1)x^k. \tag{4.29}$$

This is now written as

$$\frac{\psi(x+1)+\gamma}{x} = \sum_{k=0}^\infty \zeta(k+2)\Gamma(k+1)\frac{(-x)^j}{j!}. \tag{4.30}$$

Ramanujan's Master Theorem gives

$$\int_0^\infty x^{\nu-1}\,\frac{\psi(x+1)+\gamma}{x}\,dx = \Gamma(\nu)\Gamma(1-\nu)\zeta(2-\nu) = \frac{\pi}{\sin\pi\nu}\zeta(2-\nu).$$

The special cases $\nu = \frac{1}{2}$ and $\nu = \frac{3}{4}$ give

$$\int_0^\infty \frac{\psi(1+x)+\gamma}{x^{3/2}}\,dx = \pi\,\zeta(3/2) \tag{4.31}$$

and

$$\int_0^\infty \frac{\psi(1+x)+\gamma}{x^{5/4}}\,dx = \sqrt{2}\,\pi\,\zeta(5/4), \tag{4.32}$$

respectively.

5. A quartic integral

The authors' first encounter with Ramanujan's Master Theorem happened with the evaluation of the quartic integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{5.1}$$

The goal was to provide a proof of the experimental observation that

$$N_{0,4}(a;m) = \frac{\pi}{2^{m+3/2}\,(a+1)^{m+1/2}}P_m(a), \tag{5.2}$$

with

$$P_m(a) = 2^{-2m}\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \tag{5.3}$$

The reader will find in [2] a variety of proofs of this identity, but it was Ramanujan's Master Theorem that was the key to the first proof of (5.2). It unfolds in a story of interesting connections as is described below.

Consider a function f with the expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{\lambda(k)}{k!}(-x)^k. \tag{5.4}$$

Ramanujan's Master Theorem relates the moments of f, given by

$$A_m := \int_0^\infty x^{m-1} f(x)\,dx, \tag{5.5}$$

to the extension of the coefficients λ via the relation

$$A_m = \Gamma(m)\lambda(-m). \tag{5.6}$$

The beginning of the proof of (5.2) is an unexpected connection between the integral

$$g(c) = \int_0^\infty \frac{dx}{x^4+2ax^2+1+c} \tag{5.7}$$

and the double square root function

$$h(c) = \sqrt{a+\sqrt{1+c}} \tag{5.8}$$

given by

$$g(c) = \pi\sqrt{2}\, h'(c). \tag{5.9}$$

This leads naturally to the Taylor series expansion

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a;k-1)c^k. \tag{5.10}$$

Direct differentiation of the integral $N_{0,4}$ now yields

$$\left(\frac{d}{da}\right)^j N_{0,4}(a;k-1) = \frac{(-1)^j 2^j (k+j-1)!}{(k-1)!} \int_0^\infty \frac{x^{4k+2j-2}\,dx}{(x^4+2ax^2+1)^{k+j}}$$

and differentiation of (5.10) produces

$$\left(\frac{d}{da}\right)^j \sqrt{a+\sqrt{1+c}} = \left(\frac{d}{da}\right)^j \sqrt{a+1} + \sum_{k=1}^{\infty} \frac{\lambda(k)}{k!}(-c)^k$$

where

$$\lambda(k) = \frac{(-1)^{j+1}}{\pi\sqrt{2}}(k+j-1)!2^j \times \int_0^\infty \frac{x^{4k+2j-2}\,dx}{(x^4+2ax^2+1)^{k+j}}. \tag{5.11}$$

To evaluate $\lambda(-m)$, replace k by $-m$ in (5.11) and choose $j = 2m+1$ to arrive at

$$\lambda(-m) = \frac{m!\,2^{2m+1}}{\pi\sqrt{2}} N_{0,4}(a;m). \tag{5.12}$$

This gives the right-hand side of (5.6).

The next task is to evaluate the moments of the function

$$H(c) := \left(\frac{d}{da}\right)^j \sqrt{a+\sqrt{1+c}}, \tag{5.13}$$

which can be given as

$$A_m = \frac{(-1)^{j+1}\,(2j-3)!}{2^{2(j-1)}\,(j-2)!} \int_0^\infty \frac{c^{m-1}\,dc}{(a+\sqrt{1+c})^{j-1/2}}. \tag{5.14}$$

The key in the evaluation of $N_{0,4}$ turned out to be the fact that these moments can be computed explicitly. First, write

$$A_m = \frac{(4m-1)!}{2^{4m}\,(2m-1)!} C_m(a) \tag{5.15}$$

where

$$C_m(a) = \int_0^\infty \frac{x^{m-1}\,dx}{(a+\sqrt{1+x})^{2m+1/2}}. \tag{5.16}$$

The substitution $u = \sqrt{1+x}$ shows that

$$C_m(a) = 2\int_1^\infty f_m(u)(a+u)^{-(2m+1/2)}\,du, \tag{5.17}$$

with $f_m(u) = u(u^2-1)^{m-1}$. A closed-form for $C_m(a)$ is now obtained after integration by parts and the fact that the derivatives of f_m at $u=1$ have a closed-form expression, namely

$$C_m(a) = 2^{3m+1}\frac{m!(m-1)!(2m)!}{(4m)!(1+a)^{m+1/2}}\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \tag{5.18}$$

This gives the left-hand side of (5.6). The details can be found in [5].

6. Some classical polynomials

In this section we employ the explicit formulas for the generating functions of classical polynomials to derive some definite integrals.

6.1. The Bernoulli polynomials.

The generating function of the Bernoulli polynomials $B_m(q)$ is given by

$$\frac{te^{qt}}{e^t-1} \quad = \quad \sum_{m=0}^\infty B_m(q)\frac{t^m}{m!}. \tag{6.1}$$

These polynomials are also given in terms of the Hurwitz zeta function

$$\zeta(z,q) \quad = \quad \sum_{n=0}^\infty \frac{1}{(n+q)^z} \tag{6.2}$$

by $B_m(q) = -m\zeta(1-m,q)$ for $m \geq 1$. Then (6.1) yields

$$\frac{e^{-qt}}{1-e^{-t}} - \frac{1}{t} \quad = \quad \sum_{m=0}^\infty \zeta(-m,q)\frac{(-t)^m}{m!}. \tag{6.3}$$

Ramanujan's Master Theorem now provides the integral representation

$$\int_0^\infty t^{\nu-1}\left(\frac{e^{-qt}}{1-e^{-t}} - \frac{1}{t}\right)dt \quad = \quad \Gamma(\nu)\zeta(\nu,q), \tag{6.4}$$

valid in the range $0 < \text{Re}\,\nu < 1$.

6.2. The Hermite polynomials.

The generating function for the Hermite polynomials $H_m(x)$ is

$$e^{2xt-t^2} = \sum_{m=0}^\infty H_m(x)\frac{t^m}{m!}. \tag{6.5}$$

Their analytic continuation is given by

$$H_m(x) = 2^m U\left(-\frac{m}{2}, \frac{1}{2}, x^2\right) \tag{6.6}$$

where U is Whittaker's confluent hypergeometric function. Ramanujan's Master Theorem now provides the integral evaluation

$$\int_0^\infty t^{s-1}e^{-2xt-t^2}\,dt = \frac{\Gamma(s)}{2^s}\,U\left(\frac{s}{2},\frac{1}{2},x^2\right). \tag{6.7}$$

An equivalent form of this evaluation appears as entry 3.462.1 in [15]. The hypergeometric representation

$$H_{2m}(x) = (-1)^m\frac{(2m)!}{m!}\;{}_1F_1\left(\left.\begin{matrix}-m\\ \frac{1}{2}\end{matrix}\right| x^2\right), \tag{6.8}$$

leads to

$$\int_0^\infty t^{s-1}e^{-t^2}\cosh 2xt\,dt = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\;{}_1F_1\left(\left.\begin{matrix}s/2\\ 1/2\end{matrix}\right| x^2\right). \tag{6.9}$$

A similar integral with cosh replaced by sinh can be derived from the corresponding hypergeometric representation of H_{2m+1}.

6.3. **The Laguerre polynomials.** The Laguerre polynomials $L_n(x)$ given by

$$\frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right) \;=\; \sum_{n=0}^\infty L_n(x)t^n \tag{6.10}$$

can be expressed also as $L_n(x) = M(-n,1;x)$, where

$$M(a,c;x) = {}_1F_1\left(\left.\begin{matrix}a\\ c\end{matrix}\right| x\right) = \sum_{j=0}^\infty \frac{(a)_j}{(c)_j}\frac{x^j}{j!} \tag{6.11}$$

is the confluent hypergeometric or Kummer function. Ramanujan's Master Theorem yields the evaluation

$$\int_0^\infty \frac{t^{\nu-1}}{1+t}\exp\left(\frac{xt}{1+t}\right)\,dt \;=\; \Gamma(\nu)\Gamma(1-\nu)M(\nu,1;x). \tag{6.12}$$

The change of variables $r = t/(1+t)$ gives now

$$M(\nu,1;x) \;=\; \frac{1}{\Gamma(\nu)\Gamma(1-\nu)}\int_0^1 r^{\nu-1}(1-r)^{-\nu}e^{rx}\,dr, \tag{6.13}$$

that is entry 9.211.2 in [15].

6.4. **The Jacobi polynomials.** The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by the generating function

$$\sum_{n=0}^\infty P_n^{(\alpha,\beta)}(x)t^n = \frac{2^{\alpha+\beta}}{R^*(x,t)}(1-t+R^*(x,t))^{-\alpha}(1+t+R^*(x,t))^{-\beta}, \tag{6.14}$$

where $R^*(x,t) = \sqrt{1-2xt+t^2}$. These polynomials admit the hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+1+\alpha)}{n!\,\Gamma(1+\alpha)}\,{}_2F_1\left(n+\alpha+\beta+1,-n;1+\alpha;\frac{1-x}{2}\right). \tag{6.15}$$

Now write $R(x,t) = R^*(x,-t)$, so that $R(x,t) = \sqrt{1+2xt+t^2}$, to obtain

$$2^{\alpha+\beta} R^{-1}(1+t+R)^{-\alpha}(1-t+R)^{-\beta} = \sum_{k=0}^{\infty} \lambda(k) \frac{(-t)^k}{k!} \tag{6.16}$$

where

$$\lambda(k) = \frac{\Gamma(k+1+\alpha)}{\Gamma(1+\alpha)} {}_2F_1\left(k+\alpha+\beta+1, -k; 1+\alpha; \frac{1-x}{2}\right). \tag{6.17}$$

Ramanujan's Master Theorem produces

$$\int_0^\infty \frac{t^{\nu-1}\,dt}{R(1+t+R)^{\alpha}(1-t+R)^{\beta}} = \frac{B(\nu, 1+\alpha-\nu)}{2^{\alpha+\beta}} {}_2F_1\left(\left. \begin{matrix} 1+\alpha+\beta-\nu, \nu \\ 1+\alpha \end{matrix} \right| \frac{1-x}{2} \right).$$

6.5. **The Chebyshev polynomials.** These polynomials are defined by

$$U_n(a) = \frac{\sin((n+1)x)}{\sin x}, \quad \text{where } \cos x = a, \tag{6.18}$$

and they have the generating function

$$\sum_{k=0}^{\infty} U_k(a) x^k = \frac{1}{1-2ax+x^2}. \tag{6.19}$$

The usual application of Ramanujan's Master Theorem yields

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{1+2ax+x^2} = \frac{\pi}{\sin \pi\nu} \frac{\sin[(1-\nu)\cos^{-1} a]}{\sqrt{1-a^2}}. \tag{6.20}$$

This evaluation appears as entry 3.252.12 in [15].

7. Random Walk Integrals

In this section, we consider the n-dimensional integral

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi i x_k} \right|^s dx_1 dx_2 \cdots dx_n \tag{7.1}$$

which has recently been studied in [6] and [7]. This integral is connected to planar random walks. In detail, such a walk is said to be *uniform* if it starts at the origin and at each step takes a unit-step in a random direction. As such, (7.1) expresses the s-th moment of the distance to the origin after n steps. The study of these walks originated with K. Pearson more than a century ago [18].

For s an even integer, the moments $W_n(s)$ take integer values. In fact, for integers $k \geq 0$, the explicit formula

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2 \tag{7.2}$$

has been established in [6]. The evaluation of $W_n(s)$ for values of $s \neq 2k$ is more challenging. In particular, the definition (7.1) is not well-suited for high-precision numerical evaluations, and other representations are needed.

Here, we indicate how Ramanujan's Master Theorem may be applied to find a one-dimensional integral representation for $W_n(s)$. While (7.1) may be used to justify a priori that Ramanujan's Master Theorem 3.2 applies, it should be noted that one may proceed formally with only the sequence (7.2) given. This is the approach taken in the proof of Theorem 7.1. Ramanujan's Master Theorem produces a formal candidate for an analytic extension of the sequence $W_n(2k)$. This argument yields the following Bessel integral representation of (7.1), previously obtained by D. Broadhurst [8].

Theorem 7.1. *Let* $s \in \mathbb{C}$ *with* $2k > \text{Re } s > \max(-2, -\frac{n}{2})$. *Then*

$$W_n(s) = 2^{s+1-k} \frac{\Gamma(1+\frac{s}{2})}{\Gamma(k-\frac{s}{2})} \int_0^\infty x^{2k-s-1} \left(-\frac{1}{x}\frac{d}{dx} \right)^k J_0^n(x)\, dx. \tag{7.3}$$

Proof. The evaluation (7.2) yields the generating function for the even moments:

$$\sum_{k \geq 0} W_n(2k) \frac{(-x)^k}{(k!)^2} = \left(\sum_{k\geq 0} \frac{(-x)^k}{(k!)^2} \right)^n = J_0(2\sqrt{x})^n, \tag{7.4}$$

with $J_0(z)$ the Bessel function of the first kind as in (4.12). Applying Ramanujan's Master Theorem (1.1) to $\lambda(k) = W_n(2k)/k!$ produces

$$\Gamma(\nu)\lambda(-\nu) = \int_0^\infty x^{\nu-1} J_0^n(2\sqrt{x})\, dx. \tag{7.5}$$

A change of variables and setting $s = 2\nu$ gives

$$W_n(-s) = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_0^\infty x^{s-1} J_0^n(x)\, dx. \tag{7.6}$$

The claim now follows from the fact that if $F(s)$ is the Mellin transform of $f(x)$ then $(s-2)(s-4)\cdots(s-2k)F(s-2k)$ is the corresponding transform of $\left(-\frac{1}{x}\frac{d}{dx}\right)^k f(x)$. This follows directly from Ramanujan's Master Theorem. □

8. Extending the domain of validity

The region of validity of the identity given by Ramanujan's Master Theorem is restricted by the region of convergence of the integral. For example, the integral representation of the gamma function given in (4.4) holds for Re $s > 0$. In this section it is shown that analytic continuations of such representations are readily available by dropping the beginning of the Taylor series of the defining integrand. This provides an alternative to the method used at the end of the proof of Theorem 7.1.

Theorem 8.1. *Suppose φ satisfies the conditions of Theorem 3.2 so that for all $0 < \text{Re } s < \delta$*

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \varphi(k)(-x)^k \, dx = \frac{\pi}{\sin s\pi} \varphi(-s).$$

Then, for any positive integer N and $-N < \text{Re } s < -N+1$,

$$\int_0^\infty x^{s-1} \sum_{k=N}^\infty \varphi(k)(-x)^k \, dx = \frac{\pi}{\sin s\pi} \varphi(-s). \tag{8.1}$$

Proof. It follows from Theorem 3.2 by applying it to the function $\varphi(\cdot + N)$ that

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \varphi(k+N)(-x)^k \, dx = \frac{\pi}{\sin s\pi} \varphi(-s+N).$$

Now shift s to obtain (8.1). □

Example 8.2. Apply the result (8.1) with $N = 1$ to obtain

$$\Gamma(s) = \int_0^\infty x^{s-1} \left(e^{-x} - 1 \right) \, dx. \tag{8.2}$$

This integral representation now gives an analytic continuation of (4.4) to $-1 < \text{Re } s < 0$.

9. The method of brackets

In this section, a multi-dimensional extension of Ramanujan's Master Theorem is discussed. This has been called the *method of brackets* and it was originally presented in [14] in the context of integrals arising from Feynman diagrams. A complete description of the operational rules of the method, together with a variety of examples, was first discussed in [13]. The basic idea is the assignment of a formal symbol $\langle a \rangle$ to the divergent integral

$$\int_0^\infty x^{a-1} \, dx. \tag{9.1}$$

The rules for operating with brackets are described next. These rules employ the symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}, \tag{9.2}$$

called the *indicator of* n.

Rule 1. *The bracket expansion*

$$\frac{1}{(a_1 + a_2 + \cdots + a_r)^\alpha} = \sum_{m_1,\ldots,m_r} \phi_{m_1,\ldots,m_r} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle \alpha + m_1 + \cdots + m_r \rangle}{\Gamma(\alpha)}$$

holds. Here $\phi_{m_1,\ldots,m_r}$ is a shorthand notation for the product $\phi_{m_1} \cdots \phi_{m_r}$. If there is no possibility of confusion this will be further abridged as $\phi_{\{m\}}$. The notation $\sum_{\{m\}}$ is to be understood likewise.

Rule 2. *A series of brackets*

$$\sum_{\{n\}} \phi_{\{n\}} f(n_1, \dots, n_r) \langle a_{11} n_1 + \cdots a_{1r} n_r + c_1 \rangle \cdots \langle a_{r1} n_1 + \cdots a_{rr} n_r + c_r \rangle$$

is assigned the value

$$\frac{1}{|det(A)|} f(n_1^*, \cdots, n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*),$$

where A is the matrix of coefficients (a_{ij}) and (n_i^*) is the solution of the linear system obtained by the vanishing of the brackets. The value is not assigned if the matrix A is not invertible.

Rule 3. *In the case where a higher dimensional series has more summation indices than brackets, the appropriate number of free variables is chosen among the indices. For each such choice, Rule 2 yields a series. Those converging in a common region are added to evaluate the desired integral.*

Example 9.1. Apply the method of brackets to

$$\int_0^\infty x^{\nu-1} F(x)\, dx \tag{9.3}$$

where F has the series representation

$$F(x) = \sum_{k=0}^\infty \phi_k \lambda(k) x^k.$$

Then (9.3) can be written as the bracket series

$$\int_0^\infty x^{\nu-1} F(x)\, dx = \int_0^\infty \sum_{k=0}^\infty \phi_k \lambda(k) x^{k+\nu-1}\, dx = \sum_k \phi_k \lambda(k) \langle k + \nu \rangle .$$

Rule 2 assigns the value

$$\sum_k \phi_k \lambda(k) \langle k + \nu \rangle = \lambda(k^*) \Gamma(-k^*) \tag{9.4}$$

where k^* is the solution of $k + \nu = 0$. Thus we obtain

$$\int_0^\infty x^{\nu-1} F(x)\, dx = \lambda(-\nu) \Gamma(\nu). \tag{9.5}$$

This is precisely Ramanujan's Master Theorem as given in Theorem 3.2.

Rule 1 is a restatement of the fact that the Mellin transform of e^{-x} is $\Gamma(s)$:

$$\begin{aligned}\frac{\Gamma(s)}{(a_1+\ldots+a_r)^s} &= \int_0^\infty x^{s-1} e^{-(a_1+\ldots+a_r)x}\,dx \\ &= \int_0^\infty x^{s-1}\prod_{i=1}^{r}\sum_{m_i}\phi_{m_i}(a_i x)^{m_i}\,dx \\ &= \sum_{\{m\}}\phi_{\{m\}}a_1^{m_1}\cdots a_r^{m_r}\langle s+m_1+\cdots+m_r\rangle .\end{aligned}$$

Example 9.1 has shown that the 1-dimensional version of Rule 2 is Ramanujan's Master Theorem. A formal argument is now presented to show that the multi-dimensional version of Rule 2 follows upon iterating the one-dimensional result. The exposition is restricted to the 2-dimensional case. Consider the bracket series

$$\sum_{n_1,n_2}\phi_{n_1}\phi_{n_2}f(n_1,n_2)\langle a_{11}n_1+a_{12}n_2+c_1\rangle\langle a_{21}n_1+a_{22}n_2+c_2\rangle \tag{9.6}$$

which encodes the integral

$$\int_0^\infty\int_0^\infty\sum_{n_1,n_2}\phi_{n_1}\phi_{n_2}f(n_1,n_2)x^{a_{11}n_1+a_{12}n_2+c_1-1}y^{a_{21}n_1+a_{22}n_2+c_2-1}\,dx\,dy.$$

Substituting $(u,v)=(x^{a_{11}}y^{a_{21}},x^{a_{12}}y^{a_{22}})$ yields $\frac{dxdy}{xy}=\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}\frac{dudv}{uv}$, and hence the above integral simplifies to

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}\int_0^\infty\int_0^\infty\sum_{n_1,n_2}\phi_{n_1}\phi_{n_2}f(n_1,n_2)u^{n_1-n_1^*-1}v^{n_2-n_2^*-1}\,du\,dv.$$

Here (n_1^*,n_2^*) is the solution to $a_{11}n_1^*+a_{12}n_2^*+c_1=0$, $a_{21}n_1^*+a_{22}n_2^*+c_2=0$. Ramanujan's Master Theorem gives

$$\int_0^\infty\sum_{n_1}\phi_{n_1}f(n_1,n_2)u^{n_1-n_1^*-1}\,du=f(n_1^*,n_2)\Gamma(-n_1^*).$$

A second application of Ramanujan's Master Theorem shows that the bracket series (9.6) evaluates to

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}f(n_1^*,n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*).$$

This is Rule 2.

Example 9.2. A gamma-like higher dimensional integral. The next example illustrates the power and ease of the method of brackets for the treatment of some multidimensional integrals such as

$$\int_0^\infty\cdots\int_0^\infty\exp\left(-(x_1+\ldots+x_n)^\alpha\right)\prod_{i=1}^{n}x_i^{s_i-1}\,dx_i. \tag{9.7}$$

It should be pointed out that this class of integrals is beyond the scope of current computer algebra systems including `Mathematica` 7 and `Maple` 12.

For simplicity of exposition, the case $n = 2$ of (9.7) is considered. The general case presents no additional difficulties:

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^{s-1} y^{t-1} \exp\left(-(x+y)^\alpha\right)\, dx\, dy \\
&= \sum_j \phi_j \int_0^\infty \int_0^\infty x^{s-1} y^{t-1} (x+y)^{\alpha j}\, dx\, dy \\
&= \sum_j \phi_j \int_0^\infty \int_0^\infty x^{s-1} y^{t-1} \sum_{n,m} \phi_{n,m} x^n y^m \frac{\langle n+m-\alpha j\rangle}{\Gamma(-\alpha j)}\, dx\, dy \\
&= \sum_{j,n,m} \phi_{j,n,m} \frac{1}{\Gamma(-\alpha j)} \langle n+m-\alpha j\rangle \langle n+s\rangle \langle m+t\rangle
\end{aligned}$$

Solving the linear equations for the vanishing of the brackets gives $n^* = -s$, $m^* = -t$, and $j^* = -\frac{s+t}{\alpha}$. The determinant of the system is α, therefore the integral is

$$\frac{1}{\alpha} \frac{1}{\Gamma(-\alpha j^*)} \Gamma(-n^*)\Gamma(-m^*)\Gamma(-j^*) = \frac{1}{\alpha} \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \Gamma\left(\frac{s+t}{\alpha}\right).$$

The generalization of this result is presented as the next theorem.

Theorem 9.3.

$$\int_0^\infty \cdots \int_0^\infty \exp\left(-(x_1 + \ldots + x_n)^\alpha\right) \prod_{i=1}^n x_i^{s_i - 1}\, dx_i = \frac{1}{\alpha} \frac{\Gamma(s_1)\Gamma(s_2)\ldots\Gamma(s_n)}{\Gamma(s_1 + \ldots + s_n)} \Gamma\left(\frac{s_1 + \ldots + s_n}{\alpha}\right)$$

Remark 9.4. The correct interpretation of Rule 3 is work in-progress. The next example illustrates the difficulties associated with this question. The evaluation

$$\int_0^\infty x^{s-1} e^{-2x}\, dx = \frac{\Gamma(s)}{2^s} \tag{9.8}$$

follows directly from the bracket expansion

$$\int_0^\infty x^{s-1} e^{-2x}\, dx = \sum_n \phi_n 2^n \langle n+s\rangle$$

and Rule 2. On the other hand, rewriting the integrand as $e^{-2x} = e^{-x}e^{-x}$ and expanding it in a bracket series produces

$$\int_0^\infty x^{s-1} e^{-x} e^{-x}\, dx = \sum_{n,m} \phi_{n,m} \langle n+m+s\rangle.$$

The resulting bracket series has more summation indices than brackets. The choice of n as a free variable, gives $m^* = -n - s$ and Rule 2 produces the convergent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Gamma(n+s) = \Gamma(s)\,{}_1F_0\left(\begin{matrix} s \\ - \end{matrix} \middle| -1 \right) = \frac{\Gamma(s)}{2^s}. \tag{9.9}$$

Symmetry dictates that the choice of m as a free variable leads to the same result. Then, as formulated before, Rule 3 would yield the correct evaluation (9.8), twice.

The trouble has its origin in that the series in (9.9) has been evaluated at the boundary of its region of convergence. Rule 3 should be modified by introducing extra parameters to distinguish different regions regions of convergence. This remains to be clarified. For instance,

$$\int_0^\infty x^{s-1} e^{-Ax} e^{-Bx}\, dx = \sum_{n,m} \phi_{n,m} A^n B^m \langle n+m+s \rangle \tag{9.10}$$

which, upon choosing n respectively m as free variables, yields the two series

$$\frac{\Gamma(s)}{B^s}\,{}_1F_0\left(\begin{matrix} s \\ - \end{matrix} \middle| -\frac{A}{B} \right), \qquad \frac{\Gamma(s)}{A^s}\,{}_1F_0\left(\begin{matrix} s \\ - \end{matrix} \middle| -\frac{B}{A} \right).$$

Both series evaluate to $\Gamma(s)/(A+B)^s$, but it is now apparent that their convergence regions are different. Accordingly, they should not be added in order to obtain the value of (9.10). The original integral (9.8) appears as the limit $A, B \to 1$.

Acknowledgements. The fifth author wishes to thank the partial support of NSF-DMS 0070757. The work of the last author was partially supported, as a graduate student, by the same grant.

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