

A PROBABILISTIC INTERPRETATION OF A SEQUENCE RELATED TO NARAYANA POLYNOMIALS

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ABSTRACT. A sequence of coefficients appearing in a recurrence for the Narayana polynomials is generalized. The coefficients are given a probabilistic interpretation in terms of beta distributed random variables. The recurrence established by M. Lasalle is then obtained from a classical convolution identity. Some arithmetical properties of the generalized coefficients are also established.

1. INTRODUCTION

The Narayana polynomials

$$(1.1) \quad \mathcal{N}_r(z) = \sum_{k=1}^r N(r, k) z^{k-1}$$

with the Narayana numbers $N(r, k)$ given by

$$(1.2) \quad N(r, k) = \frac{1}{r} \binom{r}{k-1} \binom{r}{k}$$

have a large number of combinatorial properties. In a recent paper, M. Lasalle [19] established the recurrence

$$(1.3) \quad (z+1)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n \geq 1} (-z)^n \binom{r-1}{2n-1} A_n \mathcal{N}_{r-2n+1}(z).$$

The numbers A_n satisfies the recurrence

$$(1.4) \quad (-1)^{n-1} A_n = C_n + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} A_j C_{n-j},$$

with $A_1 = 1$ and $C_n = \frac{1}{n+1} \binom{2n}{n}$ the Catalan number. This recurrence is taken here as being the definition of A_n . The first few values are

$$(1.5) \quad A_1 = 1, A_2 = 1, A_3 = 5, A_4 = 56, A_5 = 1092, A_6 = 32670.$$

Lasalle [19] shows that $\{A_n : n \in \mathbb{N}\}$ is an increasing sequence of positive integers. In the process of establishing the positivity of this sequence, he contacted D. Zeilberger, who suggested the study of the related sequence

$$(1.6) \quad a_n = \frac{2A_n}{C_n},$$

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with first few values

$$(1.7) \quad a_1 = 2, a_2 = 1, a_3 = 2, a_4 = 8, a_5 = 52, a_6 = 495, a_7 = 6470.$$

The recurrence (1.4) yields

$$(1.8) \quad (-1)^{n-1}a_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{a_j}{n-j+1}.$$

This may be expressed in terms of the numbers

$$(1.9) \quad \sigma_{n,r} := \frac{2}{n} \binom{n}{r-1} \binom{n+1}{r+1}$$

that appear as entry A108838 in *OEIS* and count Dyck paths by the number of long interior inclines. The fact that $\sigma_{n,r}$ is an integer also follows from

$$(1.10) \quad \sigma_{n,r} = \binom{n-1}{r-1} \binom{n+1}{r} - \binom{n-1}{r-2} \binom{n+1}{r+1}.$$

The relation (1.8) can also be written as

$$(1.11) \quad a_n = (-1)^{n-1} \left[2 + \frac{1}{2} \sum_{j=1}^{n-1} (-1)^j \sigma_{n,j} a_j \right].$$

The original approach by M. Lasalle [19] is to establish the relation

$$(1.12) \quad (z+1)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n \geq 1} (-z)^n \binom{r-1}{2n-1} A_n(r) \mathcal{N}_{r-2n+1}(z)$$

for some coefficient $A_n(r)$. The expression

$$(1.13) \quad \mathcal{N}_r(z) = \sum_{m \geq 0} z^m (z+1)^{r-2m-1} \binom{r-1}{2m} C_m$$

given in [12], is then employed to show that $A_n(r)$ is independent of r . This is the definition of A_n given in [19]. Lasalle mentions in passing that ‘‘J. Novak observed, as empirical evidence, that the integers $(-1)^{n-1}A_n$ are precisely the (classical) cumulants of a standard semicircular random variable’’.

The goal of this paper is to revisit Lasalle’s results, provide probabilistic interpretation of the numbers A_n and to consider Zeilberger’s suggestion.

The probabilistic interpretation of the numbers A_n starts with the semicircular distribution

$$(1.14) \quad f_1(x) = \begin{cases} \frac{2}{\pi} \sqrt{1-x^2} & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let X be a random variable with distribution f_1 . Then $X_* = 2X$ satisfies

$$(1.15) \quad \mathbb{E}[X_*^r] = \begin{cases} 0 & \text{if } r \text{ is odd} \\ C_m & \text{if } r \text{ is even, with } r = 2m, \end{cases}$$

where $C_n = \frac{1}{m+1} \binom{2m}{m}$ are the Catalan numbers. The moment generating function

$$(1.16) \quad \varphi(t) = \sum_{n=0}^{\infty} \mathbb{E}[X^n] \frac{t^n}{n!}$$

is expressed in terms of the modified Bessel function of the first kind $I_\alpha(x)$ and the cumulant generating function

$$(1.17) \quad \psi(t) = \log \varphi(t) = \sum_{n=1}^{\infty} \kappa_1(n) \frac{t^n}{n!}$$

has coefficients $\kappa_1(n)$, known as the cumulants of X . The identity

$$(1.18) \quad A_n = (-1)^{n+1} \kappa_1(2n) 2^{2n},$$

is established here. Lasalle's recurrence (1.4) now follows from the convolution identity

$$(1.19) \quad \kappa(n) = \mathbb{E}[X^n] - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa(j) \mathbb{E}[X^{n-j}]$$

that holds for any pair of moments and cumulants sequences [24]. The coefficient a_n suggested by D. Zeilberger now takes the form

$$(1.20) \quad a_n = \frac{2(-1)^{n+1} \kappa_1(2n)}{\mathbb{E}[X_*^{2n}]}.$$

In this paper, these notions are extended to the case of random variables distributed according to the symmetric beta distribution

$$(1.21) \quad f_\mu(x) = \frac{1}{B(\mu + \frac{1}{2}, \frac{1}{2})} (1 - x^2)^{\mu-1/2}, \quad \text{for } |x| \leq 1, \mu > -\frac{1}{2}$$

and 0 otherwise. The semi-circular distribution is the particular case $\mu = 1$. Here $B(a, b)$ is the classical beta function defined by the integral

$$(1.22) \quad B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad \text{for } a, b > 0.$$

These ideas lead to introduce a generalization of the Narayana polynomials and these are expressed in terms of the classical Gegenbauer polynomials $C_n^{\mu+\frac{1}{2}}$. The coefficients a_n are also generalized to a family of numbers $\{a_n(\mu)\}$ with parameter μ . The special cases $\mu = 0$ and $\mu = \pm\frac{1}{2}$ are discussed in detail.

Section 2 produces a recurrence for $\{a_n\}$ from which the fact that a_n is increasing and positive are established. The recurrence comes from a relation between $\{a_n\}$ and the Bessel function $I_\alpha(x)$. Section 3 gives an expression for $\{a_n\}$ in terms of a determinant of an upper Hessenberg matrix. The standard procedure to evaluate these determinants gives the original recurrence defining $\{a_n\}$. Section 4 introduces the probabilistic interpretation of the numbers $\{a_n\}$. The cumulants of the associated random variable are expressed in terms of the Bessel zeta function. Section 5 presents the Narayana polynomials as expected values of a simple function of a semicircular random variable. These polynomials are generalized in Section 6 and they are expressed in terms of Gegenbauer polynomials. The corresponding extension of $\{a_n\}$ are presented in Section 7. The paper concludes with some arithmetical properties of $\{a_n\}$ and its generalization corresponding to the parameter $\mu = 0$. These are described in Section 8.

2. THE SEQUENCE $\{a_n\}$ IS POSITIVE AND INCREASING

In this section a direct proof of the positivity of the numbers a_n defined in (1.8) is provided. Naturally this implies $A_n \geq 0$. The analysis employs the *modified Bessel function of the first kind*

$$(2.1) \quad I_\alpha(z) := \sum_{j=0}^{\infty} \frac{1}{j!(j+\alpha)!} \left(\frac{z}{2}\right)^{2j+\alpha}.$$

Formulas for this function appear in [16].

Lemma 2.1. The numbers a_n satisfy

$$(2.2) \quad \sum_{j=1}^{\infty} \frac{(-1)^{j-1} a_j}{(j+1)!} \frac{x^{j-1}}{(j-1)!} = \frac{2}{\sqrt{x}} \frac{I_2(2\sqrt{x})}{I_1(2\sqrt{x})}.$$

Proof. The statement is equivalent to

$$(2.3) \quad \sqrt{x} I_1(2\sqrt{x}) \times \sum_{j=1}^{\infty} \frac{(-1)^{j-1} a_j}{(j+1)!} \frac{x^{j-1}}{(j-1)!} = 2I_2(2\sqrt{x}).$$

This is established by comparing coefficients of x^n on both sides and using (1.8). \square

Now change x to x^2 in Lemma 2.1 to write

$$(2.4) \quad \sum_{j=1}^{\infty} \frac{(-1)^{j-1} a_j}{(j+1)!} \frac{x^{2j-2}}{(j-1)!} = \frac{2}{x} \frac{I_2(2x)}{I_1(2x)}.$$

The classical relations

$$(2.5) \quad \frac{d}{dz} (z^{-m} I_m(z)) = z^{-m} I_{m+1}(z), \text{ and } \frac{d}{dz} (z^{m+1} I_{m+1}(z)) = z^{m+1} I_m(z)$$

give

$$(2.6) \quad I_1'(z) = I_2(z) + \frac{1}{z} I_1(z).$$

Therefore (2.4) may be written as

$$(2.7) \quad \sum_{j=1}^{\infty} \frac{(-1)^{j-1} a_j}{(j+1)!} \frac{x^{2j-2}}{(j-1)!} = \frac{1}{x} \frac{d}{dx} \log \left(\frac{I_1(2x)}{2x} \right).$$

The relations (2.5) also produce

$$(2.8) \quad \frac{d}{dz} \left(\frac{z^{m+1} I_{m+1}(z)}{z^{-m} I_m(z)} \right) = z^{2m+1} \frac{I_m^2(z) - I_{m+1}^2(z)}{I_m^2(z)}.$$

In particular,

$$(2.9) \quad \frac{d}{dz} \left(\frac{z^2 I_2(z)}{z^{-1} I_1(z)} \right) = z^3 - z^3 \frac{I_2^2(z)}{I_1^2(z)}.$$

Replacing this relation in (2.7) gives the recurrence stated next.

Proposition 2.2. The numbers a_n satisfy the recurrence

$$(2.10) \quad 2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1} \binom{n}{k+1} a_k a_{n-k}, \quad \text{for } n \geq 2,$$

with initial condition $a_1 = 1$.

Corollary 2.3. The numbers a_n are nonnegative.

Proposition 2.4. The numbers a_n satisfy

$$(2.11) \quad 4a_n = \sum_{k=1}^{n-1} \binom{n-1}{k-1} \binom{n-1}{k} a_k a_{n-k} - \sum_{k=2}^{n-2} \binom{n-1}{k-2} \binom{n-1}{k+1} a_k a_{n-k}.$$

Proof. This follows from (2.10) and the identity

$$\binom{n}{k-1} \binom{n}{k+1} = \frac{n}{2} \left[\binom{n-1}{k-1} \binom{n-1}{k} - \binom{n-1}{k-2} \binom{n-1}{k+1} \right].$$

□

Corollary 2.5. The numbers a_n are nonnegative integers. Moreover a_n is even if n is odd.

Proof. Corollary 2.3 shows $a_n > 0$. It remains to show $a_n \in \mathbb{Z}$ and to verify the parity statement. This is achieved by simultaneous induction on n .

Assume first $n = 2m + 1$ is odd. Then (1.9) shows that $\frac{1}{2}\sigma_{n,r} \in \mathbb{Z}$ and (1.11), written as

$$(2.12) \quad a_n = (-1)^{n-1} \left[2 + \sum_{r=1}^{n-1} \frac{\sigma_{n,r}}{2} a_r \right],$$

proves that $a_n \in \mathbb{Z}$. Now write (2.10) as

$$(2.13) \quad 2(2m+1)a_{2m+1} = 2 \sum_{k=1}^m \binom{2m+1}{k-1} \binom{2m+1}{k+1} a_k a_{2m+1-k}$$

and observe that either k or $2m+1-k$ is odd. The induction hypothesis shows that either a_k or a_{2m+1-k} is even. This shows a_{2m+1} is even.

Now consider the case $n = 2m$ even. If r is odd, then a_r is even; if r is even then $r-1$ is odd and $\frac{1}{2}\sigma_{n,r} \in \mathbb{Z}$ in view of the identity

$$(2.14) \quad \sigma_{n,r} = \frac{2}{r-1} \binom{n-1}{r-2} \binom{n+1}{r+1}.$$

The result follows again from (2.12). □

Corollary 2.6. The numbers A_n are nonnegative integers.

The recurrence in Proposition 2.2 is now employed to prove that $\{a_n\}$ is an increasing sequence. The first few values are 2, 1, 2, 8, 52.

Theorem 2.7. For $n \geq 3$, the inequality $a_n > a_{n-1}$ holds.

Proof. Take the terms $k = 1$ and $k = n-1$ in the sum appearing in the recurrence in Proposition (2.2) and use $a_n > 0$ to obtain

$$(2.15) \quad a_n \geq \frac{1}{2n} \left[\binom{n}{0} \binom{n}{2} a_1 a_{n-1} + \binom{n}{n-2} \binom{n}{2} a_{n-1} a_1 \right].$$

Since $a_1 = 2$ the previous inequality yields

$$(2.16) \quad a_n \geq (n-1)a_{n-1}.$$

Hence, for $n \geq 3$, this gives $a_n - a_{n-1} \geq (n-2)a_{n-1} > 0$. □

3. AN EXPRESSION IN FORMS OF DETERMINANTS

The recursion relation (1.8) expressed in the form

$$(3.1) \quad \sum_{j=1}^m (-1)^{j-1} \binom{m}{j-1} \binom{m+1}{j+1} a_j = 2m$$

is now employed to produce a system of equations for the numbers a_n by varying m through $1, 2, 3, \dots, n$. The coefficient matrix has determinant $(-1)^{\binom{n}{2}} n!$ and Cramér's rule gives

$$(3.2) \quad a_n = \frac{(-1)^{n-1}}{n!} \det \begin{pmatrix} \binom{1}{1-1} \binom{1+1}{1+1} & 0 & 0 & \dots & 0 & 2 \\ \binom{2}{1-1} \binom{2+1}{1+1} & \binom{2}{2-1} \binom{2+1}{2+1} & 0 & \dots & 0 & 4 \\ \binom{3}{1-1} \binom{3+1}{1+1} & \binom{3}{2-1} \binom{3+1}{2+1} & \binom{3}{3-1} \binom{3+1}{3+1} & \dots & 0 & 6 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{n}{1-1} \binom{n+1}{1+1} & \binom{n}{2-1} \binom{n+1}{2+1} & \binom{n}{3-1} \binom{n+1}{3+1} & \dots & \binom{n}{n-2} \binom{n+1}{n} & 2n \end{pmatrix}$$

The power of -1 is eliminated by permuting the columns to produce the matrix

$$(3.3) \quad B_n = \begin{pmatrix} 2 & \binom{1}{1-1} \binom{1+1}{1+1} & 0 & 0 & 0 \\ 4 & \binom{2}{1-1} \binom{2+1}{1+1} & \binom{2}{2-1} \binom{2+1}{2+1} & 0 & \dots \\ 6 & \binom{3}{1-1} \binom{3+1}{1+1} & \binom{3}{2-1} \binom{3+1}{2+1} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 2n & \binom{n}{1-1} \binom{n+1}{1+1} & \binom{n}{2-1} \binom{n+1}{2+1} & \binom{n}{3-1} \binom{n+1}{3+1} \dots & \binom{n}{n-2} \binom{n+1}{n} \end{pmatrix}.$$

The representation of a_n in terms of determinants is given in the next result.

Proposition 3.1. The number a_n is given by

$$(3.4) \quad a_n = \frac{\det B_n}{n!}$$

where B_n is the matrix in (3.3).

Recall that an *upper Hessenberg matrix* is one of the form

$$(3.5) \quad H_n = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} & 0 & 0 & \dots & \dots & 0 & 0 \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_{n,1} & \beta_{n,2} & \beta_{n,3} & \beta_{n,4} & \dots & \dots & \dots & \beta_{n,n-1} & \beta_{n,n} \end{pmatrix}$$

The matrix B is of this form with

$$(3.6) \quad \beta_{i,j} = \begin{cases} 2i & \text{if } 1 \leq i \leq n \quad \text{and } j = 1 \\ \binom{i}{j-2} \binom{i+1}{j} & \text{if } j-1 \leq i \leq n \quad \text{and } j > 1. \end{cases}$$

It turns out that the recurrence (1.8) used to define the numbers a_n can be recovered if one employs (3.4).

Proposition 3.2. Define α_n by

$$(3.7) \quad \alpha_n = \frac{\det B_n}{n!}$$

where B is the matrix (3.3). Then $\{\alpha_n\}$ satisfies the recursion

$$(3.8) \quad (-1)^{n-1} \alpha_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{\alpha_j}{n-j+1}$$

and the initial condition $\alpha_1 = 1$. Therefore $\alpha_n = a_n$.

Proof. For convenience define $\det H_0 = 1$. The determinant of a Hessenberg matrix satisfies the recurrence

$$(3.9) \quad \det H_n = \sum_{r=1}^n (-1)^{n-r} \beta_{n,r} \det H_{r-1} \prod_{i=r}^{n-1} \beta_{i,i+1}.$$

A direct application of (3.9) yields

$$\begin{aligned} \alpha_n &= \frac{1}{n!} \left\{ (-1)^{n-1} (2n)(n-1)! + \sum_{r=2}^n (-1)^{n-r} \binom{n}{r-2} \binom{n+1}{r} \det B_{r-1} \prod_{i=r}^{n-1} i \right\} \\ &= 2(-1)^{n-1} + \frac{1}{n!} \sum_{r=2}^n (-1)^{n-r} \binom{n}{r-2} \binom{n+1}{r} \alpha_{r-1} (n-1)! \\ &= 2(-1)^{n-1} + \sum_{r=2}^n (-1)^{n-r} \frac{1}{n} \binom{n}{r-2} \binom{n+1}{r} \alpha_{r-1} \\ &= 2(-1)^{n-1} + \sum_{r=2}^n (-1)^{n-r} \binom{n}{r-2} \binom{n+1}{r} \frac{\alpha_{r-1}}{n-r+2} \\ &= 2(-1)^{n-1} + (-1)^{n-1} \sum_{r=1}^n (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{\alpha_j}{n-j+1}. \end{aligned}$$

This is (3.8). □

Corollary 3.3. The modified Bessel function of the first kind admits a determinant expression

$$(3.10) \quad I_1(x) = x \exp \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1} \det B_j}{(j+1)! j!^2} \left(\frac{x}{2} \right)^{2j} \right).$$

Proof. This follows by integrating the identity

$$(3.11) \quad \frac{2I_2(2x)}{x I_1(2x)} = \frac{1}{x} \frac{d}{dx} \log \frac{I_1(2x)}{2x}.$$

□

4. THE PROBABILISTIC BACKGROUND: CONJUGATE RANDOM VARIABLES

This section provides the probabilistic tools required for an interpretation of the sequence A_n defined in (1.4). The specific connections are given in Section 5.

Consider a random variable X with the *symmetric beta distribution* given in (1.21). The moments of the symmetric beta distribution, given by

$$(4.1) \quad \mathbb{E}[X^n] = \frac{1}{B(\mu + \frac{1}{2}, \frac{1}{2})} \int_{-1}^1 x^n (1-x^2)^{\mu-1/2} dx,$$

vanish for n odd and for $n = 2m$ they are

$$(4.2) \quad \mathbb{E}[X^{2m}] = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 + m)} \frac{(2m)!}{2^{2m} m!}.$$

Therefore the moment generating function is

$$(4.3) \quad \varphi_\mu(t) = \mathbb{E}[e^{tX}] = \sum_{n=0}^{\infty} \mathbb{E}[X^n] \frac{t^n}{n!} = \Gamma(\mu + 1) \sum_{m=0}^{\infty} \frac{t^{2m}}{2^{2m} m! \Gamma(\mu + m + 1)}.$$

The next proposition summarizes properties of $\varphi_\mu(t)$. The first one is to recognize the series in (4.3) from (2.1). The zeros $\{j_{\mu,k}\}$ of the Bessel function of the first kind

$$(4.4) \quad J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \alpha + 1)} \left(\frac{x}{2}\right)^{2j + \alpha}$$

appear in the factorization of φ_μ in view of the relation $I_\mu(z) = e^{-\pi i \mu/2} J_\mu(iz)$.

Proposition 4.1. The moment generating function $\varphi_\mu(t)$ of a random variable $X \sim f_\mu$ is given by

$$(4.5) \quad \varphi_\mu(t) = \Gamma(\mu + 1) \left(\frac{2}{t}\right)^\mu I_\mu(t).$$

Note 4.2. The Catalan numbers C_n appear as the even-order moments of f_μ when $\mu = 1$. More precisely, if X is distributed as f_1 (written as $X \sim f_1$), then

$$(4.6) \quad \mathbb{E}[(2X)^{2n}] = C_n \text{ and } \mathbb{E}[(2X)^{2n+1}] = 0.$$

Note 4.3. The moment generating function of f_μ admits the Weierstrass product representation

$$(4.7) \quad \varphi_\mu(t) = \prod_{k=1}^{\infty} \left(1 + \frac{t^2}{j_{\mu,k}^2}\right)$$

where $\{j_{\mu,k}\}$ are the zeros of the Bessel function of the first kind J_μ .

Definition 4.4. The *cumulant generating function* is

$$\begin{aligned} \psi_\mu(t) &= \log \varphi_\mu(t) \\ &= \log \left(\sum_{n=0}^{\infty} \mathbb{E}[X^n] \frac{t^n}{n!} \right). \end{aligned}$$

The product representation of $\varphi_\mu(t)$ yields

$$\begin{aligned} \log \varphi_\mu(t) &= \sum_{k=1}^{\infty} \log \left(1 + \frac{t^2}{j_{\mu,k}^2} \right) \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{t}{j_{\mu,k}} \right)^{2n} \\ &:= \sum_{n=1}^{\infty} \kappa_\mu(n) \frac{t^n}{n!}. \end{aligned}$$

The series converges for $|t| < j_{\mu,1}$. The first Bessel zero satisfies $j_{\mu,1} > 0$ for all $\mu \geq 0$. It follows that the series has a non-zero radius of convergence.

Note 4.5. The coefficient $\kappa_\mu(n)$ is the n -th *cumulant* of X . An expression that links the moments to the cumulants of X is provided by V. P. Leonov and A. N. Shiryaev [20]:

$$(4.8) \quad \kappa_\mu(n) = \sum_{\mathcal{V}} (-1)^{k-1} (k-1)! \prod_{i=1}^k \mathbb{E}(2X)^{|V_i|}$$

where the sum is over all partitions $\mathcal{V} = \{V_1, \dots, V_k\}$ of the set $\{1, 2, \dots, n\}$.

In the case $\mu = 0$ the moments are Catalan numbers or 0, in the case $\mu = 1$ the moments are central binomial coefficients. Therefore, in both cases, the cumulants $\kappa_\mu(n)$ are integers. An expression for the general value of μ involves

$$(4.9) \quad \zeta_\mu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\mu,k}^s}$$

the *Bessel zeta function*, sometimes referred as the *Rayleigh function*.

The next result gives an expression for the cumulants of a random variable X with a distribution f_μ . The special case $\mu = 1$, described in the next section, provides the desired probabilistic interpretation of the original sequence A_n .

Theorem 4.6. Let $X \sim f_\mu$. Then

$$(4.10) \quad \kappa(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2(-1)^{n/2+1} (n-1)! \zeta_\mu(n) & \text{if } n \text{ is even.} \end{cases}$$

Proof. Rearranging the expansion in Definition 4.4 gives

$$\begin{aligned} \log \varphi_\mu(t) &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{t}{j_{\mu,k}} \right)^{2n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^{2n} \sum_{k=1}^{\infty} \frac{1}{j_{\mu,k}^{2n}}. \end{aligned}$$

Now compare powers of t in this expansion with the definition

$$(4.11) \quad \log \varphi_\mu(t) = \sum_{n=1}^{\infty} \kappa_\mu(n) \frac{t^n}{n!}$$

to obtain the result. □

The next ingredient in the search for an interpretation of the sequence A_n is the notion of conjugate random variables. The properties described below appear in [25]. A complex-valued random variable Z is called a *regular random variable* (rrv for short) if $\mathbb{E}|Z|^n < \infty$ for all $n \in \mathbb{N}$ and

$$(4.12) \quad \mathbb{E}[h(Z)] = h(\mathbb{E}[Z])$$

for all polynomials h . The class of rrv is closed under compositions with polynomials (if Z is rrv and P is a polynomial, then $P(Z)$ is rrv) and it is also closed under addition of independent rrv. The basic definition is stated next.

Definition 4.7. Let X, Y be real random variables, not necessarily independent. The pair (X, Y) is called *conjugate random variables* if $Z = X + iY$ is an rrv. The random variable X is called *self-conjugate* if Y has the same distribution as X .

The property of rrv may be expressed in terms of the function

$$\Phi(\alpha, \beta) := \mathbb{E}[\exp(i\alpha X + i\beta Y)]$$

The next theorem gives a condition for $Z = X + iY$ to be an rrv. The random variables X and Y are not necessarily independent.

Theorem 4.8. Let $Z = X + iY$ be a complex valued random variable with $\mathbb{E}[Z] = 0$ and $\mathbb{E}[Z^n] < \infty$. Then Z is an rrv if and only if $\Phi(\alpha, i\alpha) = 1$ for all $\alpha \in \mathbb{C}$.

This is now reformulated for real and independent random variables.

Theorem 4.9. Let X, Y be independent real valued random variables with finite moments. Define

$$\Phi_X(\alpha) = \mathbb{E}[e^{i\alpha X}] = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \mathbb{E}[X^n] \quad \text{and} \quad \Phi_Y(\beta) = \mathbb{E}[e^{i\beta Y}] = \sum_{n=0}^{\infty} \frac{(i\beta)^n}{n!} \mathbb{E}[Y^n].$$

Then $Z = X + iY$ is an rrv with mean zero if and only if $\Phi_X(\alpha)\Phi_Y(i\alpha) = 1$.

Example 4.10. Let X and Y be independent Gaussian variables with zero mean and the same variance. Then X and Y are conjugate since

$$\varphi_X(t) = \exp\left(-\frac{\sigma^2}{2}t^2\right) \quad \text{and} \quad \varphi_{iY}(t) = \exp\left(-\frac{\sigma^2}{2}t^2\right).$$

Note 4.11. Suppose $Z = X + iY$ is a rrv with $\mathbb{E}[Z] = 0$ and $z \in \mathbb{C}$. The condition (4.12) becomes

$$(4.13) \quad \mathbb{E}[h(z + X + iY)] = h(z).$$

Given a sequence of polynomials $\{Q_n(z)\}$ such that $\deg(Q_n) = n$ and with leading coefficient 1, an elementary argument shows that there is a unique sequence of coefficients $\alpha_{j,n}$ such that the relation

$$(4.14) \quad Q_{n+1}(z) - zQ_n(z) = \sum_{j=0}^n \alpha_{j,n} Q_j(z)$$

holds. This section discusses this recurrence for the sequence of polynomials

$$(4.15) \quad P_n(z) := \mathbb{E}(z + X)^n$$

associated to a random variable X . The polynomial P_n is of degree n and has leading coefficient 1. It is shown that if the cumulants of odd order vanish, then the even order cumulants provide the coefficients $\alpha_{j,n}$ for the recurrence (4.14).

Theorem 4.12. Let X be a random variable with cumulants $\kappa(m)$. Assume the odd-order cumulants vanish and that X has a conjugate random variable Y . Define the polynomials

$$(4.16) \quad P_n(z) = \mathbb{E}[(z + X)^n].$$

Then P_n satisfies the recurrence

$$(4.17) \quad P_{n+1}(z) - zP_n(z) = \sum_{m \geq 1} \binom{n}{2m-1} \kappa(2m) P_{n-2m+1}(z).$$

Proof. Let X_1, X_2 independent copies of X . Then

$$\begin{aligned} \mathbb{E}[X_1((X_1 + iY_1 + z + X_2)^n - (z + X_2)^n)] &= \\ &= \sum_{j=0}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + iY_1)^j (z + X_2)^{n-j}] - \mathbb{E}[X_1(z + X_2)^n]. \end{aligned}$$

This last expression becomes

$$\sum_{j=1}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + iY_1)^j (z + X_2)^{n-j}] = \sum_{j=1}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + iY_1)^j] \mathbb{E}[(z + X_2)^{n-j}].$$

On the other hand

$$\begin{aligned} \mathbb{E}[X_1((X_1 + z + X_2 + iY_1)^n - (z + X_2)^n)] &= \\ &= \sum_{r=0}^n \binom{n}{r} \mathbb{E}[X_1(X_1 + z)^{n-r}] \mathbb{E}[(X_2 + iY_1)^r] - \mathbb{E}[X_1(z + X_2)^n]. \end{aligned}$$

The cancellation property (4.28) shows that the only surviving term in the sum is $r = 0$, therefore

$$\begin{aligned} \mathbb{E}[X_1((X_1 + z + X_2 + iY_1)^n - (z + X_2)^n)] &= \\ &= \mathbb{E}[X_1(X_1 + z)^n] - \mathbb{E}[X_1] \mathbb{E}[(z + X_2)^n] \end{aligned}$$

and $\mathbb{E}[X_1] = 0$ since $\kappa(1) = 0$. This shows the identity

$$(4.18) \quad \sum_{j=1}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + iY_1)^j] \mathbb{E}[(z + X_2)^{n-j}] = \mathbb{E}[X_1(X_1 + z)^n].$$

The cumulants of X satisfy

$$(4.19) \quad \kappa(m) = \mathbb{E}X(X + iY)^{m-1}, \quad \text{for } m \geq 1,$$

(see Theorem 3.3 in [13]), therefore in the current situation

$$(4.20) \quad \mathbb{E}[X_1(X_1 + iY_1)^j] = \begin{cases} 0 & \text{if } j \text{ is even} \\ \kappa(2m) & \text{if } j = 2m + 1 \text{ is odd.} \end{cases}$$

On the other hand

$$\begin{aligned} \mathbb{E}[X_1(X_1 + z)^n] &= \mathbb{E}[(X_1 + z)^{n+1} - z(X_1 + z)^n] \\ &= P_{n+1}(z) - zP_n(z). \end{aligned}$$

Replacing in (4.18) yields the result. \square

Recall that a random variable has a Laplace distribution if its distribution function is

$$(4.21) \quad f_L(x) = \frac{1}{2}e^{-|x|}.$$

Assume X_μ has a distribution f_μ defined in (1.21) and moment generating function given by (4.7). The next lemma constructs a random variable Y_μ conjugate to X_μ .

Lemma 4.13. Let $Y_{\mu,n}$ be a random variable defined by

$$(4.22) \quad Y_{\mu,n} = \sum_{k=1}^n \frac{L_k}{j_{\mu,k}}$$

where $\{L_k : k \in \mathbb{N}\}$ is a sequence of independent, identically distributed Laplace random variables. Then $\lim_{n \rightarrow \infty} Y_{\mu,n} = Y_\mu$ exists and is a random variable with continuous probability density. Moreover, the moment generating function of iY_μ is

$$(4.23) \quad \mathbb{E}[e^{itY_\mu}] = \prod_{k=1}^{\infty} \left(1 + \frac{t^2}{j_{\mu,k}^2}\right)^{-1}.$$

the reciprocal of the moment generating function of f_μ given in (4.7).

Proof. The characteristic function of a Laplace random variable $iL_k/j_{\mu,k}$ is

$$(4.24) \quad \varphi_{iL_k}(t) = \frac{1}{1 + \frac{t^2}{j_{\mu,k}^2}}.$$

The values

$$(4.25) \quad \mathbb{E}\left[\frac{L_k}{j_{\mu,k}}\right] = 0, \text{ and } \mathbb{E}\left[\frac{L_k^2}{j_{\mu,k}^2}\right] = \frac{2}{j_{\mu,k}^2},$$

guarantee the convergence of the series

$$(4.26) \quad \sum_{k=1}^{\infty} \mathbb{E}\left[\frac{L_k}{j_{\mu,k}}\right] \text{ and } \sum_{k=1}^{\infty} \mathbb{E}\left[\frac{L_k^2}{j_{\mu,k}^2}\right].$$

(The last series evaluates to $1/(2\mu + 2)$). This ensures the existence of the limit defining Y_μ (see [17] for details). The continuity of the limiting probability density Y_μ is ensured by the fact that at least one term (in fact all) in the defining sum has a continuous probability density that is of bounded variation. \square

Note 4.14. In the case $X_\mu \sim f_\mu$ is independent of Y_μ , then the conjugacy property states that if h is an analytic function in a neighborhood \mathcal{O} of the origin, then

$$(4.27) \quad \mathbb{E}[h(z + X_\mu + iY_\mu)] = h(z), \quad \text{for } z \in \mathcal{O}.$$

In particular

$$(4.28) \quad \mathbb{E}[(X_\mu + iY_\mu)^n] = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note 4.15. In the special case $\mu = n/2 - 1$ for $n \in \mathbb{N}$, $n \geq 3$, the function (4.23) has been characterized in [11] as the moment generating function of the total time T_n spent in the sphere S^{n-1} by an n -dimensional Brownian motion starting at the origin.

5. THE NARAYANA POLYNOMIALS AND THE SEQUENCE A_n

The result of Theorem 4.12 is now applied to a random variable $X \sim f_1$. In this case the polynomials P_n correspond, up to a change of variable, to the Narayana polynomials \mathcal{N}_n . The recurrence established by M. Lasalle comes from the results in Section 4. In particular, this provides an interpretation of the sequence $\{A_n\}$ in terms of cumulants and the Bessel zeta function.

Recall the distribution function f_1

$$(5.1) \quad f_1(x) = \begin{cases} 2\sqrt{1-x^2}/\pi, & \text{for } |x| \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.1. Let $X \sim f_1$. The Narayana polynomials appear as the moments

$$(5.2) \quad \mathcal{N}_r(z) = \mathbb{E} \left[(1+z+2\sqrt{z}X)^{r-1} \right],$$

for $r \geq 1$.

Proof. The binomial theorem gives

$$\mathbb{E} \left[(1+z+2\sqrt{z}X)^{r-1} \right] = \sum_{j=0}^{r-1} \binom{r-1}{j} (z+1)^{r-1-j} z^{j/2} \mathbb{E} \left[(2X)^j \right].$$

The result now follows from (4.6) and (1.13). \square

In order to apply Theorem 4.12 consider the identities

$$(5.3) \quad \begin{aligned} \mathcal{N}_r(z) &= \mathbb{E} \left[(1+z+2\sqrt{z}X)^{r-1} \right] \\ &= (2\sqrt{z})^{r-1} \mathbb{E} \left[(X+z_*)^{r-1} \right] \\ &= (2\sqrt{z})^{r-1} P_{r-1}(z_*), \end{aligned}$$

with

$$(5.4) \quad z_* = \frac{1+z}{2\sqrt{z}}.$$

The recurrence (4.17) applied to the polynomial $P_n(z_*)$ yields

$$(5.5) \quad \frac{\mathcal{N}_{n+2}(z)}{(2\sqrt{z})^{n+1}} - \frac{(1+z)\mathcal{N}_{n+1}(z)}{2\sqrt{z}(2\sqrt{z})^n} = \sum_{m \geq 1} \binom{n}{2m-1} \kappa(2m) \frac{\mathcal{N}_{n-2m+2}(z)}{(2\sqrt{z})^{n-2m+1}}$$

that reduces to

$$(5.6) \quad (1+z)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = - \sum_{m \geq 1} \binom{r-1}{2m-1} \kappa(2m) 2^{2m} z^m \mathcal{N}_{r+1-2m}(z),$$

by using $r = n+1$. This recurrence has the form of (1.12).

Theorem 5.2. Let $X \sim f_1$. Then the coefficients A_n in Definition 1.3 are given by

$$(5.7) \quad A_n = (-1)^{n+1} \kappa(2n) 2^{2n}.$$

The expression in (4.10) gives the next result.

Corollary 5.3. Let

$$(5.8) \quad \zeta_\mu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\mu,k}^s}$$

be the Bessel zeta function. Then the coefficients A_n are given by

$$A_n = 2^{2n+1}(2n-1)! \zeta_1(2n).$$

The scaled coefficients a_n are now expressed in terms of the Bessel zeta function.

Corollary 5.4. The coefficients a_n are given by

$$(5.9) \quad a_n = 2^{2n+1}(n+1)!(n-1)! \zeta_1(2n).$$

Note 5.5. This expression for the coefficients and the recurrence

$$(5.10) \quad (n+\mu)\zeta_\mu(2n) = \sum_{r=1}^{n-1} \zeta_\mu(2r)\zeta_\mu(2n-2r).$$

given in [15], provides a new proof of the recurrence in Proposition (2.2).

6. THE GENERALIZED NARAYANA POLYNOMIALS

The Narayama polynomials $\mathcal{N}_r(z)$, defined in (1.1), have been expressed as the moments

$$(6.1) \quad \mathcal{N}_r(z) = \mathbb{E} \left[(1+z+2\sqrt{z}X)^{r-1} \right],$$

for $r \geq 1$. Here X is a random variable with distribution function f_1 . This suggests the extension

$$(6.2) \quad \mathcal{N}_n^\mu(z) = \mathbb{E} \left[(1+z+2\sqrt{z}X)^{n-1} \right],$$

with $X \sim f_\mu$. Therefore, $\mathcal{N}_n = \mathcal{N}_n^1$.

Note 6.1. The same argument given in (5.6) gives the recurrence

$$(6.3) \quad (1+z)\mathcal{N}_r^\mu(z) - \mathcal{N}_{r+1}^\mu(z) = - \sum_{m \geq 1} \binom{r-1}{2m-1} \kappa(2m) 2^{2m} z^m \mathcal{N}_{r+1-2m}^\mu(z),$$

where $\kappa(2n)$ are the cumulants of $X \sim f_\mu$. Theorem 5.2 gives an expression for the generalization of the Lasalle numbers:

$$(6.4) \quad A_n^\mu := (-1)^{n+1} \kappa(2n) 2^{2n}$$

and the corresponding expression in terms of the Bessel zeta function:

$$(6.5) \quad A_n^\mu := 2^{2n+1}(2n-1)! \zeta_\mu(2n).$$

The generalized Narayana polynomials are now expressed in terms of the Gegenbauer polynomials $C_n^\mu(x)$ defined by the generating function

$$(6.6) \quad \sum_{n=0}^{\infty} C_n^\mu(x)t^n = (1-2xt+t^2)^{-\mu}.$$

These polynomial admit several hypergeometric representations:

$$\begin{aligned}
(6.7) \quad C_n^\mu(x) &= \frac{(2\mu)_n}{n!} {}_2F_1\left(-n, n+2\mu; \mu + \frac{1}{2}; \frac{1-x}{2}\right) \\
&= \frac{2^n (\mu)_n}{n!} (x-1)^n {}_2F_1\left(-n, -n-\mu + \frac{1}{2}; -2n-2\mu+1; \frac{2}{1-x}\right) \\
&= \frac{(2\mu)_n}{n!} \left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n, -n-\mu + \frac{1}{2}; \mu + \frac{1}{2}; \frac{x-1}{x+1}\right).
\end{aligned}$$

The connection between Narayana and Gegenbauer polynomials comes from the expression for $C_n^\mu(z)$ given in the next proposition.

Proposition 6.2. The Gegenbauer polynomials are given by

$$(6.8) \quad C_n^\mu(z) = \frac{(2\mu)_n}{n!} \mathbb{E} \left[\left(z + \sqrt{z^2 - 1} X_{\mu-1/2} \right)^n \right].$$

Proof. The Laplace integral representation

$$(6.9) \quad C_n^\mu(\cos \theta) = \frac{\Gamma(n+2\mu)}{2^{2\mu-1} n! \Gamma^2(\mu)} \int_0^\pi (\cos \theta + i \sin \theta \cos \phi)^n \sin^{2\mu-1} \phi d\phi$$

appears as Theorem 6.7.4 in [3]. The change of variables $z = \cos \theta$ and $X = \cos \phi$ gives

$$\begin{aligned}
C_n^\mu(z) &= \frac{\Gamma(n+2\mu)}{2^{2\mu} n! \Gamma^2(\mu)} \int_{-1}^1 \left(z + \sqrt{z^2 - 1} X \right)^n (1-X^2)^{\mu-1} dX \\
&= \frac{(2\mu)_n}{n!} \mathbb{E} \left[\left(z + \sqrt{z^2 - 1} X_{\mu-1/2} \right)^n \right],
\end{aligned}$$

as claimed. Since this is a polynomial identity in z , it can be extended to all $z \in \mathbb{C}$. \square

Theorem 6.3. The Gegenbauer polynomial C_n^μ and the generalized polynomial \mathcal{N}_n^μ satisfy the relation

$$(6.10) \quad \mathcal{N}_{n+1}^\mu(z) = \frac{n!}{(2\mu+1)_n} (1-z)^n C_n^{\mu+\frac{1}{2}} \left(\frac{1+z}{1-z} \right).$$

Proof. Introduce the variable

$$(6.11) \quad Z = \frac{1+z}{1-z}$$

so that

$$(6.12) \quad z = \frac{Z-1}{Z+1} \quad \text{and} \quad \frac{Z}{\sqrt{Z^2-1}} = \frac{1+z}{2\sqrt{z}}.$$

Then

$$\begin{aligned}
C_n^{\mu+\frac{1}{2}} \left(\frac{1+z}{1-z} \right) &= \frac{(2\mu+1)_n}{n!} \left(\frac{2\sqrt{z}}{1-z} \right)^n \mathbb{E} \left[\left(\frac{1+z}{2\sqrt{z}} + X_\mu \right)^n \right] \\
&= \frac{(2\mu+1)_n}{n! (1-z)^n} \mathbb{E} \left[(1+z + 2\sqrt{z} X_\mu)^n \right] \\
&= \frac{(2\mu+1)_n}{n! (1-z)^n} \mathcal{N}_{n+1}^\mu(z),
\end{aligned}$$

using $Z^2 - 1 = 4z/(1-z)^2$. \square

The expression (6.7) now provides hypergeometric expressions for the original Narayana polynomials

$$(6.13) \quad \mathcal{N}_{n+1}(z) = \frac{2(1-z)^n}{(n+2)(n+1)} C_n^{3/2} \left(\frac{1+z}{1-z} \right).$$

Corollary 6.4. The Narayana polynomials are given by

$$(6.14) \quad \begin{aligned} \mathcal{N}_{n+1}(z) &= (1-z)^n {}_2F_1 \left(-n, n+3; 2; \frac{z}{z-1} \right) \\ &= \frac{(2n+2)!}{(n+2)!(n+1)!} z^n {}_2F_1 \left(-n, -n-1; -2n-2; \frac{z-1}{z} \right) \\ &= {}_2F_1(-n, -n-1; 2; z). \end{aligned}$$

This yields the representation as finite sums

$$(6.15) \quad \begin{aligned} \mathcal{N}_{n+1}(z) &= \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} \binom{n+k+2}{k} z^k (1-z)^{n-k} \\ &= \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} \binom{2n+2-k}{n-k} z^{n-k} (1-z)^k \\ &= \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k+1} \binom{n+1}{k} z^k. \end{aligned}$$

Note that the first two expressions coincide up to the change of summation variable $k \rightarrow n-k$ while the third identity is nothing but (1.1).

Note 6.5. The representation

$$(6.16) \quad C_n^\mu(z) = \frac{(\mu)_n}{n!} (2x)^n {}_2F_1 \left(-\frac{n}{2}, \frac{1-n}{2}; 1-n-\mu; \frac{1}{x^2} \right)$$

that appears in as 6.4.12 in [3], gives the expression

$$(6.17) \quad \mathcal{N}_{n+1}(z) = \frac{(2n+2)!}{(n+1)!(n+2)!} \left(\frac{1+z}{2} \right)^n {}_2F_1 \left(-\frac{n}{2}, \frac{1-n}{2}; -n-\frac{1}{2}; \left(\frac{1-z}{1+z} \right)^2 \right)$$

equal to the finite sum representation

$$(6.18) \quad \mathcal{N}_{n+1}(z) = \frac{1}{2^{n-1}(n+2)} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n+1-2k}{n-2k} (1-z)^{2k} (1+z)^{n-2k}.$$

Note 6.6. The polynomials $S_n(z) = z\mathcal{N}_n^1(z)$ satisfy the symmetry identity

$$(6.19) \quad S_n(z) = z^{n+1} S_n(z^{-1}).$$

These polynomials were expressed in [21] as

$$(6.20) \quad S_n(z) = (z-1)^{n+1} \int_0^{z/(z-1)} P_n(2x-1) dx$$

where $P_n(x) = C_n^{1/2}(x)$ are the Legendre polynomials. An equivalent formulation is provided next.

Theorem 6.7. The polynomials $S_n(z)$ are given by

$$S_n(z) = \frac{1}{2^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n+1-k} \binom{2n-2k}{n-k} \binom{n+1-k}{k} (z-1)^{2k} (z+1)^{n+1-2k}.$$

Proof. The integration rule

$$(6.21) \quad \int C_n^\mu(x) dx = \frac{1}{2(\mu-1)} C_{n+1}^{\mu-1}(x)$$

implies

$$(6.22) \quad \int_0^{z/(z-1)} C_n^{1/2}(2x-1) dx = -\frac{1}{2} C_{n+1}^{-1/2} \left(\frac{z+1}{z-1} \right),$$

since the generating function

$$(6.23) \quad \sum_{n=0}^{\infty} t^n C_n^{-1/2}(z) = (1-2zt+t^2)^{1/2}$$

gives $C_{n+1}^{-1/2}(-1) = 0$ for $n > 1$. Then (6.20) yields

$$(6.24) \quad S_n(z) = -\frac{1}{2} (z-1)^{n+1} C_{n+1}^{-1/2} \left(\frac{z+1}{z-1} \right).$$

A classical formula for the Gegenbauer polynomials states

$$(6.25) \quad C_n^\mu(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!} \frac{(\mu)_{n-k}}{(n-2k)!} (2z)^{n-2k}$$

and the identity

$$\left(-\frac{1}{2} \right)_k = -\frac{1}{2^{2k-1}} \frac{(2k-2)!}{(k-1)!}$$

produce

$$(6.26) \quad C_n^{-1/2}(z) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1}}{n-k} \binom{2n-2k-2}{n-k-1} \binom{n-k}{k} z^{n-2k}.$$

The result now follows from (6.24). \square

7. THE GENERALIZATION OF THE NUMBERS a_n

The terms forming the original suggestion of Zeilberger

$$(7.1) \quad a_n = \frac{2A_n}{C_n}$$

have been given a probabilistic interpretation: let X be a random variable with a symmetric beta distribution function with parameter $\mu = 1$ given explicitly in (5.1). The numerator A_n is

$$(7.2) \quad A_n = (-1)^{n+1} \kappa(2n) 2^{2n}$$

where $\kappa(2n)$ is the even-order cumulant of the scaled random variable $X_* = 2X$. The denominator C_n is interpreted as the even-order moment of X_* :

$$(7.3) \quad C_n = \mathbb{E} [X_*^{2n}].$$

These notions are used now to define an extension of the coefficients a_n .

Definition 7.1. Let X be a random variable with vanishing odd cumulants. The numbers $a_n(\mu)$ are defined by

$$(7.4) \quad a_n(\mu) = \frac{2(-1)^{n+1}\kappa(2n)}{\mathbb{E}[X_*^{2n}]}$$

In the special case $X_* = 2X$ with $X \sim f_\mu$, these numbers are computed using the cumulants

$$(7.5) \quad \kappa_\mu(2n) = (-1)^{n+1}2^{2n+1}(2n-1)!\zeta_\mu(2n)$$

and the even order moments

$$(7.6) \quad \mathbb{E}[X_*^{2n}] = \frac{(2n)!}{n!} \frac{1}{(\mu+1)_n}$$

to produce

$$(7.7) \quad a_n(\mu) = 2^{2n+1}(n-1)!(\mu+1)_n \zeta_\mu(2n).$$

The value

$$(7.8) \quad \zeta_\mu(2) = \frac{1}{4(\mu+1)}$$

yields the initial condition $a_1(\mu) = 2$.

The recurrence (5.10) now provides the next result. Recall that when x is not necessarily a positive integer, the binomial coefficient is given by

$$(7.9) \quad \binom{x}{k} = \frac{\Gamma(x+1)}{\Gamma(x-k+1)k!}.$$

Proposition 7.2. The coefficients $a_n(\mu)$ satisfy the recurrence

$$(7.10) \quad a_n(\mu) = \frac{1}{2\binom{n+\mu-1}{n-1}} \sum_{k=1}^{n-1} \binom{n+\mu-1}{n-k-1} \binom{n+\mu-1}{k-1} a_k(\mu) a_{n-k}(\mu),$$

with initial condition $a_1(\mu) = 2$.

Proof. Start with the convolution identity for Bessel zeta functions (5.10) and replace each zeta function by its expression in terms of $a_n(\mu)$ from (7.7), which gives

$$(n+\mu) \frac{a_n(\mu)}{2^{2n+1}(n-1)!(\mu+1)_n} = \sum_{k=1}^{n-1} \frac{a_k(\mu)}{2^{2k+1}(k-1)!(\mu+1)_k} \frac{a_{n-k}(\mu)}{2^{2n-2k+1}(n-k-1)!(\mu+1)_{n-k}}$$

and after simplification

$$a_n(\mu) = \frac{1}{2(n+\mu)} \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k-1)!} \frac{(\mu+1)_n}{(\mu+1)_k(\mu+1)_{n-k}} a_k(\mu) a_{n-k}(\mu).$$

The result now follows by elementary algebra. \square

Note 7.3. In the case $\mu = 1$, the recurrence (7.10) becomes (2.10) and the coefficients $a_n(1)$ are the original numbers a_n .

Note 7.4. The recurrence (7.10) can be written as

$$a_n(\mu) = \frac{1}{2} \sum_{k=1}^{n-1} \frac{\Gamma(n)\Gamma(\mu+1)\Gamma(n+\mu)}{\Gamma(\mu+k+1)\Gamma(n+\mu-k+1)\Gamma(n-k)\Gamma(k)} a_k(\mu)a_{n-k}(\mu).$$

Theorem 7.5. The coefficients $a_n(\mu)$ are positive and increasing for $n \geq \lfloor \frac{\mu+3}{2} \rfloor$.

Proof. The positivity is clear from (7.7). Now take the terms corresponding to $k=1$ and $k=n-1$ in (7.10) to obtain

$$(7.11) \quad a_n(\mu) \geq \frac{n-1}{\mu+1} a_1(\mu)a_{n-1}(\mu) = \frac{2(n-1)}{\mu+1} a_{n-1}(\mu).$$

This yields

$$(7.12) \quad a_n(\mu) - a_{n-1}(\mu) \geq \frac{2n-3-\mu}{\mu+1} a_{n-1}(\mu)$$

and the result follows. \square

Some other special cases are considered next.

The case $\mu=0$. In this situation the distribution is the arcsine distribution given by

$$(7.13) \quad f_0(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, & \text{for } |x| \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

By the recurrence on the ζ_0 function, the coefficients

$$(7.14) \quad a_n(0) = 2^{2n}(n-1)!n!\zeta_0(2n)$$

satisfy the recurrence

$$(7.15) \quad a_n(0) = \frac{1}{2} \sum_{k=1}^{n-1} \binom{n-1}{k} \binom{n-1}{k-1} a_k(0)a_{n-k}(0)$$

with $a_1(0) = 2$. Now define as Lasalle $b_n = \frac{1}{2}a_n(0)$ and then (7.15) becomes

$$(7.16) \quad \begin{aligned} b_n &= \sum_{k=1}^{n-1} \binom{n-1}{k} \binom{n-1}{k-1} b_k b_{n-k}, \\ b_1 &= 1. \end{aligned}$$

In particular b_n is a positive integer.

The following comments are obtained by an analysis similar to that for a_n .

Note 7.6. The recurrence

$$\sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \binom{n-1}{j-1} b_j = 1$$

gives the generating function

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} b_j}{j!} \frac{x^{2j-2}}{(j-1)!} = \frac{I_1(2x)}{x I_0(2x)} = \frac{1}{2x} \frac{d}{dx} \log I_0(2x).$$

Note 7.7. The sequence b_n admits a determinant representation $b_n = \det(M_n)$, where

$$(7.17) \quad M_n = \begin{pmatrix} 1 & \binom{1}{1} \binom{1-1}{1-1} & 0 & 0 & \cdots & 0 \\ 1 & \binom{2}{1} \binom{2-1}{1-1} & \binom{2}{2} \binom{2-1}{2-1} & 0 & \cdots & 0 \\ 1 & \binom{3}{1} \binom{3-1}{1-1} & \binom{3}{2} \binom{3-1}{2-1} & \binom{3}{3} \binom{3-1}{3-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{n}{1} \binom{n-1}{1-1} & \binom{n}{2} \binom{n-1}{2-1} & \binom{n}{3} \binom{n-1}{3-1} & \cdots & \binom{n}{n-1} \binom{n-1}{n-2} \end{pmatrix}$$

Note 7.8. The identity $I_2(x) = I_0(x) - \frac{2}{x}I_1(x)$ is expressed as

$$(7.18) \quad \frac{I_1(2x)}{xI_0(2x)} \left[1 + \frac{1}{2}x^2 \frac{2I_2(2x)}{xI_1(2x)} \right] = 1$$

provides the relation

$$(7.19) \quad b_n = \frac{1}{2} \sum_{j=1}^{n-1} \binom{n-1}{j} \binom{n}{j-1} b_j a_{n-j}.$$

The case $\mu = \frac{1}{2}$. In this situation the distribution is the uniform distribution on $[-1, 1]$ with even moments

$$(7.20) \quad \mathbb{E}X_*^{2n} = \frac{2^{2n}}{2n+1}$$

and vanishing odd moments. The sequence of cumulants is

$$(7.21) \quad \kappa_{1/2}(2n) = 2(-1)^{n+1}(2n-1)! \zeta_{1/2}(2n)$$

where the Bessel zeta function is

$$(7.22) \quad \zeta_{1/2}(2n) = \sum_{k=1}^{\infty} \frac{1}{\pi^{2n} k^{2n}} = \frac{1}{\pi^{2n}} \zeta(2n) = \frac{2^{2n-1}}{(2n)!} |B_{2n}|,$$

where B_n are the Bernoulli numbers. This follows from the identity

$$(7.23) \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

This yields

$$(7.24) \quad \kappa_{1/2}(2n) = 2^{2n} \frac{B_{2n}}{2n} \text{ and } \kappa_{1/2}(2n+1) = 0,$$

with $\kappa_{1/2}(0) = 0$. These are the coefficients of $u^n/n!$ in the cumulant moment generating function

$$(7.25) \quad \log \varphi_{1/2}(u) = \log \frac{\sinh u}{u} = \frac{1}{6}u^2 - \frac{1}{180}u^4 + \frac{1}{2835}u^6 + \cdots.$$

Finally, the corresponding sequence

$$(7.26) \quad a_n \left(\frac{1}{2} \right) = \frac{2(-1)^{n+1} \kappa(2n)}{\mathbb{E}[X_*^{2n}]}$$

is given by

$$(7.27) \quad a_n \left(\frac{1}{2} \right) = 2^{2n} \frac{2n+1}{n} |B_{2n}|.$$

The first few terms are

$$(7.28) \quad a_1\left(\frac{1}{2}\right) = 2, a_2\left(\frac{1}{2}\right) = \frac{4}{3}, a_3\left(\frac{1}{2}\right) = \frac{32}{9}, a_4\left(\frac{1}{2}\right) = \frac{96}{5}, a_5\left(\frac{1}{2}\right) = \frac{512}{3},$$

as expected, this is an increasing sequence for $n \geq 3$. The convolution identity (5.10) for Bessel zeta functions gives the well-known quadratic relation for the Bernoulli numbers

$$(7.29) \quad \sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1)B_{2n}, \quad \text{for } n > 1.$$

Moreover, the moment-cumulants relation (1.19) gives, replacing n by $2n$ and after simplification, the other well-known identity

$$(7.30) \quad \sum_{j=1}^n \binom{2n+1}{2j} 2^{2j} B_{2j} = 2n, \quad \text{for } n \geq 1.$$

Note 7.9. The generating function of the sequence $a_n\left(\frac{1}{2}\right)$ is given by

$$\frac{I_{3/2}(x)}{xI_{1/2}(x)} = \frac{x \tanh x - 1}{x^2} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2a_j\left(\frac{1}{2}\right)}{(2j+1)(2j-1)!} x^{2j-2}.$$

The limiting case $\mu = -\frac{1}{2}$ has the probability distribution

$$(7.31) \quad f_{-1/2}(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$$

(the discrete Rademacher distribution). For a Rademacher random variable X , the odd moments of $X_* = 2X$ vanish while the even order moments are

$$(7.32) \quad \mathbb{E}[X_*^{2n}] = 2^{2n}.$$

Therefore

$$(7.33) \quad \kappa_{-1/2}(2n) = (-1)^{n+1} 2^{2n+1} (2n-1)! \zeta_{-1/2}(2n).$$

The identity

$$(7.34) \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

shows that $j_{k,-1/2} = (2k-1)\pi/2$ and therefore

$$(7.35) \quad \zeta_{-1/2}(2n) = \sum_{k=1}^{\infty} \frac{2^{2n}}{\pi^{2n} (2k-1)^{2n}} = \frac{2^{2n}-1}{\pi^{2n}} \zeta(2n).$$

The expression for $\kappa_{-1/2}(2n)$ may be simplified by the relation

$$(7.36) \quad E_n = -\frac{2}{n+1} (2^{n+1}-1) B_{n+1}$$

between the Euler numbers E_n and the Bernoulli numbers. It follows that

$$(7.37) \quad \kappa_{-1/2}(2n) = -2^{4n-1} E_{2n-1}.$$

The corresponding sequence $a_n\left(-\frac{1}{2}\right)$ is now given by

$$(7.38) \quad a_n\left(-\frac{1}{2}\right) = (-1)^n 2^{2n} E_{2n-1}$$

and its first few values are

$$a_1\left(-\frac{1}{2}\right) = 2, a_2\left(-\frac{1}{2}\right) = 4, a_3\left(-\frac{1}{2}\right) = 32, a_4\left(-\frac{1}{2}\right) = 544, a_5\left(-\frac{1}{2}\right) = 15872,$$

Note 7.10. The generating function of the sequence $a_n \left(-\frac{1}{2}\right)$ is given by

$$\frac{I_{1/2}(x)}{xI_{-1/2}(x)} = \frac{\tanh x}{x} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2a_j \left(-\frac{1}{2}\right)}{(2j-1)!} x^{2j-2}.$$

Note 7.11. The convolution identity (5.10) yields the well-known quadratic recurrence relation

$$(7.39) \quad \sum_{k=1}^{n-1} \binom{2n-2}{2k-1} E_{2k-1} E_{2n-2k-1} = 2E_{2n-1}, \text{ for } n > 1,$$

and the moment-cumulant relation (1.19) gives the other well-known identity

$$(7.40) \quad \sum_{k=1}^n \binom{2n-1}{2k-1} 2^{2k-1} E_{2k-1} = 1, \text{ for } n \geq 1.$$

8. SOME ARITHMETIC PROPERTIES OF THE SEQUENCES a_n AND b_n

Given a sequence of integers $\{x_n\}$ it is often interesting to examine its arithmetic properties. For instance, given a prime p , this is measured by the p -adic valuation $\nu_p(x_n)$, defined as the largest power of p that divides x_n . Examples of this process appear in [2] for the Stirling numbers and in [1, 22] for a sequence of coefficients arising from a definite integral.

The statements described below give information about $\nu_p(a_n)$. These results will be presented in a future publication. M. Lasalle [19] established the next theorem by showing that A_n and C_n have the same parity. The fact that the Catalan numbers are odd if and only if $n = 2^r - 1$ for some $r \geq 2$ provides the proof. This result appears in [14, 18].

Theorem 8.1. The integer a_n is odd if and only if $n = 2(2^m - 1)$.

The previous statement may be expressed in terms of the sequence of binary digits of n .

Experimental Fact 8.2. Let $B(n)$ be the binary digits of n and denote \bar{x} a sequence of a arbitrary length consisting of the repetitions of the symbol x . The following statements hold (experimentally)

- 1) $\nu_2(a_n) = 0$ if and only if $B(n) = \{\bar{1}, 0\}$.
- 2) $\nu_2(a_n) = 1$ if and only if $B(n) = \{\bar{1}\}$ or $\{1, \bar{0}\}$.
- 3) $\nu_2(a_n) = 2$ if and only if $B(n) = \{1, 0, \bar{1}, 0\}$.

The experimental findings for the prime $p = 3$ are described next.

Experimental Fact 8.3. Suppose n is not of the form $3^m - 1$. Then

$$(8.1) \quad \nu_3(a_{3n-2}) = \nu_3(a_{3n-1}) = \nu_3(a_{3n}).$$

Define $w_j = 3^j - 1$. Suppose n lies in the interval $w_j + 1 \leq n \leq w_{j+1} - 1$. Then

$$(8.2) \quad \nu_3(a_{3n+2}) = j - \nu_3(n + 1).$$

If $n = w_j$, then $\nu_3(a_{3n}) = 0$.

Now assume that $n = 3^m - 1$. Then

$$(8.3) \quad \nu_3(a_{3n}) = \nu_3(a_{3n-1}) - 1 = \nu_3(a_{3n-2}) - 1 = m.$$

Experimental Fact 8.4. The last observation deals with the sequence $\{a_n(\mu)\}$. Consider it now as defined by the recurrence (7.10). The initial condition $a_1(\mu) = 2$, motivated by the origin of the sequence, in general does not provide integer entries. For example, if $\mu = 2$, the sequence is

$$\left\{ 2, \frac{2}{3}, \frac{8}{9}, \frac{7}{3}, \frac{88}{9}, \frac{1594}{27}, \frac{1448}{3} \right\},$$

and for $\mu = 3$

$$\left\{ 2, \frac{1}{2}, \frac{1}{2}, \frac{39}{40}, 3, \frac{263}{20}, \frac{309}{4} \right\}.$$

Observe that the denominators of the sequence for $\mu = 2$ are always powers of 3, but for $\mu = 3$ the arithmetic nature of the denominators is harder to predict. On the other hand if in the case $\mu = 3$ the initial condition is replaced by $a_1(3) = 4$, then the resulting sequence has denominators that are powers of 5. This motivates the next definition.

Definition 8.5. Let x_n be a sequence of rational numbers and p be a prime. The sequence is called p -integral if the denominator of x_n is a power of p .

Therefore if $a_1(3) = 4$, then the sequence $a_n(3)$ is 5-integral. The same phenomena appears for other values of μ , the data is summarized in the next table.

μ	2	3	4	5	6	7	8	9
$a_1(\mu)$	2	4	10	12	84	264	990	2860
p	3	5	7	7	11	11	13	17

Note 8.6. The sequence $\{2, 4, 10, 12, 84, 264, 990\}$ does not appear in Sloane's sequences list OEIS.

This suggests the next conjecture.

Conjecture 8.7. Let $\mu \in \mathbb{N}$. Then there exists an initial condition $a_1(\mu)$ and a prime p such that the sequence $a_n(\mu)$ is p -integral.

Some elementary arithmetical properties of a_n are discussed next. A classical result of E. Lucas states that a prime p divides the binomial coefficient $\binom{a}{b}$ if and only if at least one of the base p digits of b is greater than the corresponding digit of a .

Proposition 8.8. Assume n is odd. Then a_n is even.

Proof. Let $n = 2m + 1$. The recurrence (2.10) gives

$$\begin{aligned} 2(2m+1)a_{2m+1} &= \sum_{k=1}^{2m} \binom{2m+1}{k-1} \binom{2m+1}{k+1} a_k a_{2m+1-k} \\ &= 2 \sum_{k=1}^m \binom{2m+1}{k-1} \binom{2m+1}{k+1} a_k a_{2m+1-k}. \end{aligned}$$

For k in the range $1 \leq k \leq m$, one of the indices k or $2m+1-k$ is odd. The induction argument shows that for each such k , either a_k or a_{2m+1-k} is an even integer. This completes the argument. \square

Lemma 8.9. Assume $n = 2^m - 1$. Then $\frac{1}{2}a_n$ is an odd integer.

Proof. Proposition 8.8 shows that $\frac{1}{2}a_n$ is an integer. The relation (1.8) may be written as

$$(8.4) \quad (-1)^{n-1}a_n = 2 + \frac{1}{n} \sum_{j=1}^{n-1} (-1)^j \binom{n}{j-1} \binom{n+1}{j+1} a_j.$$

This implies

$$(8.5) \quad n [(-1)^{n-1} \frac{1}{2}a_n - 1] = \frac{1}{2} \sum_{j=1}^{n-1} (-1)^j \binom{n}{j-1} \binom{n+1}{j+1} a_j.$$

Observe that if j is odd, then a_j is even and $\binom{n+1}{j+1}$ is also even. Therefore the corresponding term in the sum is divisible by 4. If j is even, then Lucas's theorem shows that 4 divides $\binom{n+1}{j+1}$. It follows that the right hand side is an even number. This implies that $\frac{1}{2}a_n$ is odd, as claimed. \square

The next statement, which provides the easier part of Theorem 8.1, describes the indices that produce odd values of a_n .

Theorem 8.10. If $n = 2(2^m - 1)$, then a_n is odd.

Proof. Isolate the term $j = n/2$ in the identity (8.4) to produce

$$\begin{aligned} [(-1)^n a_n + 2] (2^m - 1) &= \binom{2^{m+1} - 2}{2^m - 2} \binom{2^{m+1} - 1}{2^m} \frac{1}{2} a_{n/2} \\ &+ \frac{1}{2} \sum_{j \neq n/2} (-1)^j \binom{n}{j-1} \binom{n+1}{j+1} a_j. \end{aligned}$$

Lemma 8.9 shows that $\frac{1}{2}a_{n/2}$ is odd and the binomial coefficients on the first term of the right-hand side are also odd by Lucas' theorem. Each term of the sum is even because a_j is even if j is odd and for j even $\binom{n}{j-1}$ is even. Therefore the entire right-hand side is even which forces a_n to be odd. \square

The final result discussed here deals with the parity of the sequence b_n . The main tool is the recurrence

$$(8.6) \quad b_n = \sum_{k=1}^{n-1} \binom{n-1}{k} \binom{n-1}{k-1} b_k b_{n-k}$$

with $b_1 = 1$. Observe that the binomial coefficients appearing in this recurrence are related to the Narayana numbers $N(n, k)$ (1.2) by

$$(8.7) \quad \binom{n-1}{k} \binom{n-1}{k-1} = (n-1)N(n-1, k-1).$$

Arithmetic properties of the Narayana numbers have been discussed by M. Bona and B. Sagan [5]. It is established that if $n = 2^m - 1$ then $N(n, k)$ is odd for $0 \leq k \leq n - 1$; while if $n = 2^m$ then $N(n, k)$ is even for $1 \leq k \leq n - 2$.

The next theorem is the analog of M. Lasalle's result for the sequence b_n .

Theorem 8.11. The coefficient b_n is an odd integer if and only if $n = 2^m$, for some $m \geq 0$.

Proof. The first few terms $b_1 = 1$, $b_2 = 1$, $b_3 = 4$ support the base case of an inductive proof.

If n is odd, then

$$(8.8) \quad b_n = (n-1) \sum_{k=1}^{n-1} N(n-1, k-1) b_k b_{n-k}$$

shows that b_n is even.

Consider now the case $n = 2^m$. Then Lucas' theorem shows that $\binom{2^m-1}{k} \binom{2^m-1}{k-1}$ is odd for all k . The inductive step states that b_k is even if $k \neq 2^r$. In the case $k = 2^r$, then b_{n-k} is odd if and only if $k = 2^{m-1}$, in which case all the terms in (8.8) are even with the single exception $\binom{2^m-1}{2^{m-1}} \binom{2^m-1}{2^{m-1}-1} b_{2^{m-1}}^2$. This shows that b_n is odd.

Finally, if $n = 2j$ is even with $j \neq 2^r$, then

$$(8.9) \quad b_n = \binom{2j-1}{j} \binom{2j-1}{j-1} b_j^2 + 2 \sum_{k=1}^{j-1} \binom{n-1}{k} \binom{n-1}{k-1} b_k b_{n-k}.$$

Now simply observe that $j \neq 2^r$, therefore b_j is even by induction. It follows that b_n itself is even.

This completes the proof. \square

9. ONE FINAL QUESTION

Sequences of combinatorial origin often turn out to be unimodal or logconcave. Recall that a sequence $\{x_j : 1 \leq j \leq n\}$ is called *unimodal* if there is an index m_* such that $x_1 \leq x_2 \leq \dots \leq x_{m_*}$ and $x_{m_*+1} \geq x_{m_*+2} \geq \dots \geq x_n$. The sequence is called *logconcave* if $x_{n+1}x_{n-1} \geq x_n^2$. An elementary argument shows that a logconcave sequence is always unimodal. The reader will find in [4, 6, 7, 8, 9, 10, 23, 26] a variety of examples of these type of sequences.

Conjecture 9.1. The sequences $\{a_n\}$ and $\{b_n\}$ are logconcave.

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