

# GENERALIZED BERNOULLI NUMBERS AND A FORMULA OF LUCAS

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ABSTRACT. An overlooked formula of E. Lucas for the generalized Bernoulli numbers is proved using generating functions. This is then used to provide a new proof and a new form of a sum involving classical Bernoulli numbers studied by K. Dilcher. The value of this sum is then given in terms of the Meixner-Pollaczek polynomials.

## 1. INTRODUCTION

The goal of this paper is to provide a unified approach to two topics that have appeared in the literature. The first one is an expression for the generalized Bernoulli numbers  $B_n^{(p)}$  defined by the exponential generating function

$$(1.1) \quad \sum_{n=0}^{\infty} B_n^{(p)} \frac{z^n}{n!} = \left( \frac{z}{e^z - 1} \right)^p.$$

For  $n \in \mathbb{N}$ , the coefficients  $B_n^{(p)}$  are polynomials in  $p$  named after Nörlund in [1]. The first few are

$$(1.2) \quad B_0^{(p)} = 1, B_1^{(p)} = -\frac{1}{2}p, B_2^{(p)} = -\frac{1}{12}p + \frac{1}{4}p^2, B_3^{(p)} = \frac{1}{8}p^2(1 - p).$$

In his 1878 paper E. Lucas [4] gave the formula

$$(1.3) \quad B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!} \frac{n!}{(n-p)!} \beta^{n-p+1} (1 + \beta) \cdots (p-1 + \beta)$$

for  $n \geq p$ . This is a symbolic formula: to obtain the value of  $B_n^{(p)}$ , expand the expression (1.3) and replace  $\beta^j$  by the ratio  $B_j/j$ . Here  $B_j$  is the classical Bernoulli number  $B_n = B_n^{(1)}$  in the notation from (1.1).

The second topic is an expression established by K. Dilcher [2] for the sums of products of Bernoulli numbers

$$(1.4) \quad S_N(n) := \sum \binom{2n}{2j_1, 2j_2, \dots, 2j_N} B_{2j_1} B_{2j_2} \cdots B_{2j_N},$$

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where the sum is taken over all nonnegative integers  $j_1, \dots, j_N$  such that  $j_1 + \dots + j_N = n$ , and where

$$(1.5) \quad \binom{2n}{2j_1, 2j_2, \dots, 2j_N} = \frac{(2n)!}{(2j_1)! \cdots (2j_N)!}$$

is the multinomial coefficient and  $B_{2k}$  is the classical Bernoulli number. One of the main results of [2] is the evaluation

$$(1.6) \quad S_N(n) = \frac{(2n)!}{(2n-N)!} \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} b_k^{(N)} \frac{B_{2n-2k}}{2n-2k},$$

where the coefficients  $b_k^{(N)}$  are defined by the recurrence

$$(1.7) \quad b_k^{(N+1)} = -\frac{1}{N} b_k^{(N)} + \frac{1}{4} b_{k-1}^{(N-1)},$$

with  $b_0^{(1)} = 1$  and  $b_k^{(N)} = 0$  for  $k < 0$  and for  $k > \lfloor (N-1)/2 \rfloor$ .

Lucas's original proof is recalled in Section 2. This section also contains an extension of Lucas's formula for  $B_n^{(p)}$  to  $0 \leq n \leq p-1$  in terms of the Stirling numbers of the first kind. A unified proof of the two formulas for  $B_n^{(p)}$  based on generating functions is given in Section 3. Another proof of Lucas's formula, based on recurrences, is given in Section 4 and Section 5 contains a proof of

$$(1.8) \quad S_N(n) = \sum_{k=0}^N \frac{(2n)!}{(2n-k)!} 2^{-k} \binom{N}{k} B_{2n-k}^{(N-k)}$$

that expresses Dilcher's sum (1.4) explicitly in terms of the generalized Bernoulli numbers. Expressing this result in hypergeometric form leads to a formula for  $S_N(n)$  in terms of the Meixner-Pollaczek polynomials

$$(1.9) \quad P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{n\phi} {}_2F_1 \left( \begin{matrix} -n & \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\phi} \right).$$

It is then established that the recurrence (1.7), provided by Dilcher in [2], is equivalent to the classical three-term relation for this orthogonal family of polynomials.

## 2. LUCAS'S THEOREM

In his paper [4], E. Lucas gave an expression for the generalized Bernoulli numbers  $B_n^{(p)}$ , for  $n \geq p$ . This section presents an outline of his proof and an extension of this expression for  $B_n^{(p)}$  to the case  $0 \leq n \leq p-1$ . A proof based on generating functions is given in the next section. Lucas's formula uses the translation

$$(2.1) \quad \beta^n = \frac{B_n}{n}$$

coming from umbral calculus. Observe, for example, that

$$\begin{aligned} B_3^{(2)} &= \frac{(-1)^1 3!}{1! 1!} \beta^2 (1 + \beta) = -6(\beta^2 + \beta^3) \\ &= -6 \left( \frac{B_2}{2} + \frac{B_3}{3} \right) = -3B_2 = -\frac{1}{2} \end{aligned}$$

Observe also that the symbolic substitution (2.1) should be performed only *after* all the terms have been expanded. For example,

$$(2.2) \quad \beta^2(1 + \beta) = \beta^2 + \beta^3 = \frac{B_2}{2} + \frac{B_3}{3} = -\frac{1}{4}$$

but

$$(2.3) \quad \beta^2(1 + \beta) \neq \frac{B_2}{2} \left( 1 + \frac{B_1}{1} \right) = \frac{1}{24}.$$

**Theorem 2.1** (Lucas). *For  $n \geq p$ , the generalized Bernoulli numbers  $B_n^{(p)}$  are given by*

$$(2.4) \quad B_n^{(p)} = \frac{(-1)^{p-1} n!}{(p-1)! (n-p)!} \beta^{n-p+1} (1 + \beta)(2 + \beta) \cdots (p-1 + \beta)$$

where, in symbolic notation,

$$(2.5) \quad \beta^n = \frac{B_n}{n}.$$

*Proof.* Lucas's argument begins with the identity

$$(2.6) \quad pB_n^{(p+1)} = (p-n)B_n^{(p)} - pnB_{n-1}^{(p)}$$

which follows directly from the identity for generating functions

$$(2.7) \quad x \frac{d}{dx} \left( \frac{x}{e^x - 1} \right)^p = p(1-x) \left( \frac{x}{e^x - 1} \right)^p - p \left( \frac{x}{e^x - 1} \right)^{p+1}.$$

Shifting  $n$  to  $n-1$  it follows that

$$(2.8) \quad pB_{n-1}^{(p+1)} = (p-n+1)B_{n-1}^{(p)} - p(n-1)B_{n-2}^{(p)}.$$

Now multiplying (2.6) by  $n(p+1)$  and (2.8) by  $(p-n+1)$  leads to

$$\begin{aligned} p(p+1)B_n^{(p+2)} &= (p-n+1)(p-n)B_n^{(p)} - (p-n+1)(p+p+1)nB_{n-1}^{(p)} \\ &\quad + p(p+1)n(n-1)B_{n-2}^{(p)} \end{aligned}$$

and then, by the same methods, he produces

$$\begin{aligned} (p+2)(p+1)pB_n^{(p+3)} &= (p-n+2)(p-n+1)(p-n)B_n^{(p)} \\ &\quad - (p-n_2)(p-n+1)(p+p+1+p+2)nB_{n-1}^{(p)} \\ &\quad + (p-n+2)(p(p+1) + p(p+2) + (p+1)(p+2))n(n-1)B_{n-2}^{(p)} \\ &\quad - p(p+1)(p+2)n(n-1)(n-2)B_{n-3}^{(p)} \end{aligned}$$

and then, stating '*and so on*', concludes the proof. □

The following alternate proof of Lucas's theorem using generating functions requires an expression for  $B_n^{(p)}$  in the range  $0 \leq n \leq p-1$ , of the kind given in (2.4). This cannot be obtained directly from (2.4). The Stirling numbers of the first kind  $s_k^{(p)}$  are used to produce an equivalent formulation of  $B_n^{(p)}$ . These numbers are defined by the generating function

$$(2.9) \quad z(z-1)(z-2)\cdots(z-(p-1)) = \sum_{k=1}^p s_k^{(p)} z^k.$$

Then (2.4) may be written as

$$\begin{aligned} B_n^{(p)} &= \frac{(-1)^{p-1}}{(p-1)!} n(n-1)\cdots(n-(p-1)) \beta^{n-p} (-1)^p \sum_{k=1}^p s_k^{(p)} (-\beta)^k \\ &= -\frac{1}{(p-1)!} n(n-1)\cdots(n-(p-1)) \sum_{k=1}^p s_k^{(p)} (-1)^k \frac{B_{n-p+k}}{n-p+k}. \end{aligned}$$

Observe that the index  $n$  varies in the range  $0 \leq n \leq p-1$ , therefore the prefactor  $n(n-1)\cdots(n-(p-1))$  always vanishes. On the other hand, all the summands are finite, except when  $k = p-n$ . Performing the translation from  $(-\beta)^k$  to  $B_k/k$  for this specific index gives

$$-\frac{1}{(p-1)!} n(n-1)\cdots 1 \times (-1)(-2)\cdots(-(p-1-n)) s_{p-n}^{(p)} (-1)^{p-n} = \frac{s_{p-n}^{(p)}}{\binom{p-1}{n}}.$$

This gives:

**Theorem 2.2.** *The generalized Bernoulli numbers  $B_n^{(p)}$ , with  $0 \leq n \leq p-1$  are given by*

$$(2.10) \quad B_n^{(p)} = \frac{s_{p-n}^{(p)}}{\binom{p-1}{n}}.$$

In fact, this is a classical result. It is, for example, a direct consequence of the identity

$$(2.11) \quad (z-1)(z-2)\cdots(z-p) = \sum_{\ell=0}^p \binom{p}{\ell} z^\ell B_{p-\ell}^{(p+1)}$$

which appears (unnumbered) in [5, p.149].

### 3. THE PROOF VIA GENERATING FUNCTION

The expressions for the generalized Bernoulli numbers given in (2.4) and (2.10) are now used to compute the generating function

$$(3.1) \quad G(z) = \sum_{n=0}^{\infty} B_n^{(p)} \frac{z^n}{n!}$$

and to show that it coincides with the generating function of the generalized Bernoulli numbers (1.1).

Split the sum as  $G(z) = G_1(z) + G_2(z)$ , where

$$(3.2) \quad G_1(z) = \sum_{n=0}^{p-1} B_n^{(p)} \frac{z^n}{n!} \text{ and } G_2(z) = \sum_{n=p}^{\infty} B_n^{(p)} \frac{z^n}{n!}.$$

Observe that

$$\begin{aligned} G_2(z) &= \sum_{n=p}^{\infty} \frac{(-1)^{p-1}}{(p-1)!} \frac{n!}{(n-p)!} \beta^{n-p+1} (1+\beta) \cdots ((p-1)+\beta) \frac{z^n}{n!} \\ &= \frac{(-1)^{p-1}}{(p-1)!} \beta (1+\beta) \cdots (p-1+\beta) \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} \beta^{n-p} \frac{z^n}{n!} \\ &= \frac{(-1)^{p-1}}{(p-1)!} (-1)^p \sum_{k=1}^p s_k^{(p)} (-1)^k z^p f_k(z) \end{aligned}$$

with

$$(3.3) \quad f_k(z) = \sum_{n=p}^{\infty} \frac{B_{n-p+k}}{(n-p)!(n-p+k)!} z^{n-p}.$$

The  $(k-1)$ -st antiderivative of  $f_k(z)$ , denoted by  $g_k(z)$ , is

$$\begin{aligned} g_k(z) &= \sum_{n=p}^{\infty} \frac{B_{n-p+k}}{(n-p+k)!} z^{n-p+k-1} \\ &= z^{-1} \sum_{\ell=k}^{\infty} \frac{B_{\ell}}{\ell!} z^{\ell} \\ &= \frac{1}{z} \left[ \frac{z}{e^z - 1} - \sum_{\ell=0}^{k-1} \frac{B_{\ell}}{\ell!} z^{\ell} \right], \end{aligned}$$

therefore

$$\begin{aligned} f_k(z) &= \left( \frac{d}{dz} \right)^{k-1} \frac{1}{e^z - 1} - \left( \frac{d}{dz} \right)^{k-1} \frac{1}{z} \\ &= \left( \frac{d}{dz} \right)^{k-1} \frac{1}{e^z - 1} + \frac{(-1)^k (k-1)!}{z^k}. \end{aligned}$$

This gives

$$\begin{aligned} G_2(z) &= -\frac{z^p}{(p-1)!} \sum_{k=1}^p s_k^{(p)} (-1)^k f_k(z) \\ &= -\frac{z^p}{(p-1)!} \sum_{k=1}^p s_k^{(p)} (-1)^k \left( \frac{d}{dz} \right)^{k-1} \left[ \frac{1}{e^z - 1} \right] - \frac{z^p}{(p-1)!} \sum_{k=1}^p s_k^{(p)} \frac{(k-1)!}{z^k}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
G_1(z) &= \sum_{n=0}^{p-1} B_n^{(p)} \frac{z^n}{n!} \\
&= \sum_{n=0}^{p-1} \frac{s_{p-n}^{(p)}}{\binom{p-1}{n}} \frac{z^n}{n!} \\
&= \frac{1}{(p-1)!} \sum_{n=0}^{p-1} s_{p-n}^{(p)} (p-1-n)! z^n \\
&= \frac{1}{(p-1)!} \sum_{k=1}^p s_k^{(p)} (k-1)! z^{p-k}.
\end{aligned}$$

This sum cancels the second term in the expression for  $G_2(z)$ . Hence

$$(3.4) \quad G(z) = G_1(z) + G_2(z) = -\frac{z^p}{(p-1)!} \sum_{k=1}^p s_k^{(p)} (-1)^k \left(\frac{d}{dz}\right)^{k-1} \left[\frac{1}{e^z - 1}\right].$$

Using (2.9) this gives

$$(3.5) \quad G(z) = -\frac{(-z)^p}{(p-1)!} \left( (p-1) + \frac{d}{dz} \right) \cdots \left( 1 + \frac{d}{dz} \right) \left[ \frac{1}{e^z - 1} \right].$$

The next lemma simplifies this expression. Its proof by induction is elementary, so it is omitted.

**Lemma 3.1.** *For  $n \geq 1$ , the identity*

$$(3.6) \quad \frac{(-1)^n}{n!} \left( n + \frac{d}{dz} \right) \left( n-1 + \frac{d}{dz} \right) \cdots \left( 1 + \frac{d}{dz} \right) \frac{1}{e^z - 1} = \frac{1}{(e^z - 1)^{n+1}}$$

*holds.*

Replacing in (3.5) produces

$$(3.7) \quad G(z) = -\frac{(-z)^p}{(p-1)!} \frac{(p-1)!}{(-1)^{p-1}} \frac{1}{(e^z - 1)^p} = \left( \frac{z}{e^z - 1} \right)^p,$$

which is the generating function of the generalized Bernoulli numbers. This proves both Lucas's formula for  $B_n^{(p)}$  with  $n \geq p$  and the expression (2.10) for  $0 \leq p \leq n-1$ .

#### 4. LUCAS'S FORMULA VIA RECURRENCES

The numbers  $B_n^{(p)}$  satisfy the recurrence

$$(4.1) \quad pB_n^{(p+1)} = (p-n)B_n^{(p)} - pnB_{n-1}^{(p)}.$$

Lucas's formula for  $B_n^{(p)}$  is now established by showing that the numbers defined by (2.4) satisfy the same recurrence.

Start with

$$(p-n)B_n^{(p)} - pnB_{n-1}^{(p)} = (p-n)\frac{(-1)^{p-1}n!}{(p-1)!(n-p)!}\beta^{n-p}\prod_{k=0}^{p-1}(k+\beta) - pn\frac{(-1)^{p-1}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}\prod_{k=0}^{p-1}(k+\beta),$$

and write it as

$$\begin{aligned} (p-n)B_n^{(p)} - pnB_{n-1}^{(p)} &= \frac{(-1)^{p-1}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}\left[-\prod_{k=0}^{p-1}(k+\beta) - p\beta\prod_{k=0}^{p-1}(k+\beta)\right] \\ &= \frac{(-1)^pn!}{(p-1)!(n-p-1)!}\beta^{n-1-p}(p+\beta)\prod_{k=0}^{p-1}(k+\beta) \\ &= p\frac{(-1)^p}{p!}\frac{n!}{(n-p-1)!}\beta^{n-1-p}\prod_{k=0}^p(k+\beta) \\ &= pB_n^{(p+1)}. \end{aligned}$$

To conclude the result, it suffices to check that the initial conditions match. This is clear, since

$$(4.2) \quad B_n^{(1)} = \frac{n!}{(n-1)!}\beta^n = n\beta^n = n\frac{B_n}{n} = B_n.$$

This establishes Lucas's formula for the generalized Bernoulli numbers.

### 5. A NEW PROOF OF DILCHER'S FORMULA

This section analyzes the sum

$$(5.1) \quad S_N(n) := \sum \binom{2n}{2j_1, 2j_2, \dots, 2j_N} B_{2j_1} B_{2j_2} \cdots B_{2j_N},$$

using Lucas's expression for the generalized Bernoulli numbers  $B_n^{(p)}$ . An alternative formulation is presented.

**Proposition 5.1.** *The sum  $S_N(n)$  is given by*

$$(5.2) \quad S_N(n) = \sum_{k=0}^N \frac{(2n)!}{(2n-k)!} 2^{-k} \binom{N}{k} B_{2n-k}^{(N-k)}$$

for  $2n > N$ .

*Proof.* The umbral method [7] shows that the sum  $S_N(n)$  is given by

$$(5.3) \quad S_N(n) = \frac{1}{2^N} (\epsilon_1 B_1 + \cdots + \epsilon_N B_N)^{2n}$$

with  $\epsilon_j = \pm 1$ . Introduce the notation

$$(5.4) \quad Y_{2n}^{(M,N)} = (-B_1 - \cdots - B_M + B_{M+1} + \cdots + B_N)^{2n}$$

where there are  $M$  minus signs and  $N - M$  plus signs. Thus,

$$(5.5) \quad S_N(n) = \frac{1}{2^N} \sum_{M=0}^N \binom{N}{M} Y_{2n}^{(M,N)}.$$

The next step uses the famous umbral identity

$$(5.6) \quad f(-B) = f(B) + f'(0)$$

(see Section 2 of [3] for details) to obtain

$$(5.7) \quad Y_{2n}^{(M,N)} = Y_{2n}^{(M-1,N)} + 2nY_{2n-1}^{(M-1,N-1)}.$$

This may be written as

$$(5.8) \quad Q_{2n}^{(M)} = Q_{2n}^{(M-1)} + 2nQ_{2n-1}^{(M-1)},$$

where  $Q_j^M = Y_j^{(M,P+j)}$  and  $P = N - 2n$ . Then (5.8) is easily solved to produce

$$(5.9) \quad Q_{2n}^{(M)} = \sum_{k=0}^M \binom{M}{k} \frac{(2n)!}{(2n-k)!} Q_{2n-k}^{(0)}.$$

Since the initial condition is

$$(5.10) \quad Q_{2n-k}^{(0)} = Y_{2n-k}^{(0,N-k)} = B_{2n-k}^{(N-k)},$$

it follows that

$$(5.11) \quad Y_{2n}^{(M,N)} = \sum_{k=0}^M \binom{M}{k} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)}.$$

Replacing in (5.5) yields

$$\begin{aligned} S_N(n) &= \frac{1}{2^N} \sum_{M=0}^N \binom{N}{M} Y_{2n}^{(M,N)} \\ &= \frac{1}{2^N} \sum_{M=0}^N \binom{N}{M} \sum_{k=0}^M \binom{M}{k} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)} \\ &= \frac{1}{2^N} \sum_{k=0}^N \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)} \sum_{M=0}^N \binom{M}{k} \binom{N}{M}. \end{aligned}$$

Now use the basic identity

$$(5.12) \quad \sum_{M=0}^N \binom{M}{k} \binom{N}{M} = \sum_{M=k}^N \binom{M}{k} \binom{N}{M} = 2^{N-k} \binom{N}{k}$$

to obtain the result.  $\square$

Lucas's identity for generalized Bernoulli numbers is now used to obtain a second expression for the sum  $S_N(n)$ .



**Proposition 5.2.** *For  $2n > N$ , the sum  $S_N(n)$  is given by*

$$(5.13) \quad S_N(n) = \frac{(2n)!}{(2n-N)!} \beta^{2n-N+1} \sum_{\ell=0}^{N-1} \binom{N}{\ell+1} \frac{(-1)^\ell (\beta+1)_\ell}{2^{N-1-\ell} \ell!}.$$

*Proof.* Using the Pochhammer symbol

$$(5.14) \quad (\beta+1)_{p-1} = \frac{\Gamma(\beta+p)}{\Gamma(\beta+1)} = (\beta+1) \cdots (\beta+p-1)$$

Lucas's formula (2.4) is stated in the form

$$(5.15) \quad B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!} \frac{n!}{(n-p)!} \beta^{n-p+1} (\beta+1)_{p-1}.$$

Using Proposition 5.1 and  $B_n^{(0)} = \delta_n$  so that  $B_{2n-N}^{(0)} = 0$  since  $2n > N$ , it follows that

$$\begin{aligned} S_N(n) &= \sum_{k=0}^{N-1} \frac{(2n)!}{(2n-k)!} 2^{-k} \binom{N}{k} \frac{(-1)^{N-k-1}}{(N-k-1)!} \frac{(2n-k)!}{(2n-N)!} \beta^{2n-N+1} (\beta+1)_{N-k-1} \\ &= \frac{(2n)!}{(2n-N)!} \beta^{2n-N+1} \sum_{k=0}^{N-1} 2^{-k} \binom{N}{k} \frac{(-1)^{N-k-1}}{(N-k-1)!} (\beta+1)_{N-k-1} \end{aligned}$$

that reduces to the stated form. □

To obtain a hypergeometric form of the sum  $S_N(n)$ , observe that

$$(5.16) \quad N(1-N)_\ell = (-1)^\ell \frac{N!}{(N-\ell-1)!}$$

and  $(2)_\ell = (\ell+1)!$  give

$$(5.17) \quad (-1)^\ell \binom{N}{\ell+1} = N \frac{(1-N)_\ell}{(2)_\ell},$$

and the following result follows from Proposition 5.2.

**Proposition 5.3.** *The hypergeometric form of the sum  $S_N(n)$  is given by*

$$(5.18) \quad S_N(n) = \frac{(2n)!}{(2n-N)!} \beta^{2n-N+1} 2^{1-N} N {}_2F_1 \left( \begin{matrix} 1-N, & 1+\beta \\ & 2 \end{matrix} \middle| 2 \right).$$

The final form of the sum  $S_N(n)$  involves the Meixner-Pollaczek polynomials defined by

$$(5.19) \quad P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1 \left( \begin{matrix} -n, & \lambda + ix \\ & 2\lambda \end{matrix} \middle| 1 - e^{-2i\phi} \right).$$

Choosing  $\lambda = 1$  and  $\phi = \pi/2$  gives the next result.

**Theorem 5.1.** *The sum  $S_N(n)$  is given by*

$$(5.20) \quad S_N(n) = \frac{(2n)!}{(2n-N)!} \frac{1}{(2i)^{N-1}} \beta^{2n-N+1} P_{N-1}^{(1)} \left( -i\beta; \frac{\pi}{2} \right).$$

Some examples are presented next.

**Example 5.4.** *The Meixner-Pollaczek polynomial*

$$(5.21) \quad P_2^{(1)}\left(x; \frac{\pi}{2}\right) = 2x^2 - 1$$

gives

$$\begin{aligned} S_3(n) &= \frac{(2n)!}{(2n-3)!} \times (-1/4)\beta^{2n-2}(-2\beta^2 - 1) \\ &= \frac{(2n)(2n-1)(2n-2)}{4} \left[ 2\frac{B_{2n}}{2n} + \frac{B_{2n-2}}{2n-2} \right] \\ &= (2n-1)(n-1)B_{2n} + \frac{1}{2}n(2n-1)B_{2n-2}, \end{aligned}$$

which coincides with [2, eq. (2.6)].

**Example 5.5.** *The Meixner-Pollaczek of degree 3 is*

$$(5.22) \quad P_3^{(1)}\left(x; \frac{\pi}{2}\right) = \frac{4}{3}(-2x + x^3)$$

that produces

$$\begin{aligned} S_3(n) &= \frac{(2n)!}{(2n-4)!} \frac{1}{(2i)^3} \beta^{2n-3} \frac{4}{3} (2i\beta + i\beta^3) \\ &= -\frac{1}{3}(2n-1)(n-1)(2n-3)B_{2n} - \frac{1}{3}(2n)(2n-1)(2n-3)B_{2n-2}, \end{aligned}$$

which coincides with [2, eq. (2.7)].

The next step is to establish a correspondence between the *Dilcher coefficients*  $b_k^{(N)}$  in (1.6) and the coefficients  $p_k^{(n)}$  in

$$(5.23) \quad P_n^{(1)}(x; \pi/2) = \sum_{k=0}^n p_k^{(n)} x^k$$

the Meixner-Pollaczek polynomials. In particular, it is shown that the recurrence (1.7) is a consequence of the classical three terms recurrence for orthogonal polynomials.

**Theorem 5.2.** *The coefficients  $b_k^{(N)}$  defined in (1.6) and the coefficients  $p_k^{(n)}$  are related by*

$$(5.24) \quad b_k^{(N)} = \frac{(-1)^{N-1-k}}{2^{N-1}} p_{N-1-2k}^{(N-1)}.$$

The recurrence relation (1.7) is equivalent to the three-terms recurrence

$$(5.25) \quad (n+1)P_{n+1}^{(1)}\left(x; \frac{\pi}{2}\right) - 2xP_n^{(1)}\left(x; \frac{\pi}{2}\right) + (n+1)P_{n-1}^{(1)}\left(x; \frac{\pi}{2}\right) = 0.$$

satisfied by the Meixner-Pollaczek polynomials.

*Proof.* The Meixner-Pollaczek polynomials are orthogonal, hence they satisfy a three-terms recurrence. The specific form for this family in (5.25) appears in [6, Chapter 18]. In terms of its coefficients  $p_k^{(n)}$  this is expressed as

$$(5.26) \quad (n+1)p_k^{(n+1)} - 2p_{k-1}^{(n)} + (n+1)p_k^{(n-1)} = 0.$$

Comparing the two expressions for  $S_N(n)$  in (1.6) and (5.20) gives (5.24). This is equivalent to

$$(5.27) \quad p_\ell^{(N-1)} = 2^{N-1} t^{N-1+\ell} b_{\frac{1}{2}(N-1-\ell)}^{(N)}.$$

Replacing in (5.26) and simplifying yields (1.7).  $\square$

Theorem 2 in [2], stated below, may be proven along the same lines of the proof of Theorem 2.2. Details are omitted.

**Theorem 5.3.** *If  $2n \leq N - 1$ , then*

$$(5.28) \quad \begin{aligned} S_N(n) &= (-1)^n \frac{(2n)!(N-2n-1)!}{2^{N-1}} p_{N-2n-1}^{(N-1)} \\ &= (-1)^{N-1} (2n)!(N-2n-1)! b_n^{(N)}. \end{aligned}$$

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