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On Some Integrals Involving the Hurwitz Zeta Function: Part 1*

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Abstract. We establish a series of integral formulae involving the Hurwitz zeta function. Applications are given to integrals of Bernoulli polynomials, $\ln\Gamma(q)$ and $\ln\sin(q)$.

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1. Introduction

The Hurwitz zeta function, defined by

$$\zeta(z,q)=\sum_{n=0}^{\infty}\frac{1}{(n+q)^z} \tag{1.1}$$

for $z\in\mathbb{C}$ and $q\neq 0,\,-1,\,-2,\ldots,$ is one of the fundamental transcendental functions. The series converges for $\text{Re}(z)>1$ so that $\zeta(z,q)$ is an analytic function of z in this region. The integral representation

$$\zeta(z,q)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{e^{-qt}}{1-e^{-t}}t^{z-1}dt, \tag{1.2}$$

where $\Gamma(z)$ is Euler's gamma function, is valid for $\text{Re}(z)>1$ and $\text{Re}(q)>0$, and can be used to show that $\zeta(z,q)$ admits an analytic extension to the whole complex plane except for a simple pole at $z=1$. In most of the examples discussed here we consider only the range $0<q\leq 1$. Special cases of $\zeta(z,q)$ include the Riemann zeta function

$$\zeta(z,1)=\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z} \tag{1.3}$$

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and

$$\zeta\left(z, \frac{1}{2}\right) = 2^z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^z} = (2^z - 1)\zeta(z). \tag{1.4}$$

The function $\zeta(z, q)$ admits several integral representations in addition to (1.2). For example, Hermite proved

$$\zeta(z, q) = \frac{1}{2} q^{-z} + \frac{1}{z-1} q^{1-z} + 2q^{1-z} \int_0^{\infty} \frac{\sin(z \tan^{-1} t)\, dt}{(1+t^2)^{z/2}(e^{2\pi t q} - 1)}, \tag{1.5}$$

which is valid for $q > 0$ and $z \neq 1$. In fact, (1.5) is an explicit representation of the analytic continuation of (1.1) to $\mathbb{C} - \{1\}$.

Among the many places in which $\zeta(z, q)$ appears we mention the evaluation by Kölbig [18] of integrals of the form

$$R_m(\mu, \nu) = \int_0^{\infty} e^{-\mu t} t^{\nu - 1} \ln^m t \, dt, \tag{1.6}$$

an example of which is

$$R_2(\mu, \nu) = \mu^{-\nu} \Gamma(\nu)[(\psi(\nu) - \ln \mu)^2 + \zeta(2, \nu)]. \tag{1.7}$$

Here

$$\psi(x) = \Gamma'(x)/\Gamma(x) \tag{1.8}$$

is the logarithmic derivative of $\Gamma(x)$, also called the digamma function.

The Hurwitz zeta function also plays a role in Vardi's evaluations [29] of

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)\sqrt{2\pi}}{\Gamma(1/4)} \right) \tag{1.9}$$

and [28] of Kinkelin's constant

$$\ln A := \lim_{k \to \infty} \left[\ln(1^1 2^2 \ldots k^k) - \frac{1}{2}(k^2 + k + 1) \ln k + \frac{k^2}{4} \right] \tag{1.10}$$

as

$$\ln A = \exp\left(\frac{1}{12} - \zeta'(-1) \right). \tag{1.11}$$

Yue and Williams [32, 33] established the integral representation

$$\zeta(z, q) = 2(2\pi)^{z-1} \int_0^{\infty} \frac{e^x \sin(\pi z/2 + 2\pi q) - \sin(\pi z/2)}{e^{2x} - 2e^x \cos 2\pi q + 1} x^{-z} dx \tag{1.12}$$

and used it to evaluate definite integrals, (1.9) among them. For example, for $0 < a < 1$, they obtain

$$\int_0^\infty \frac{e^{-x}\ln x\,dx}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1} = \frac{\pi}{2\sin 2\pi a}\ln\left(\frac{\Gamma(1-a)}{(2\pi)^{2a-1}\,\Gamma(a)}\right).$$

Integrals involving the Hurwitz zeta function also appear in problems dealing with distributions of $\{nx\}$ for $x \notin \mathbb{Q}$ and $n \in \mathbb{N}$, where $\{x\}$ denotes the fractional part of x. In this context Mikolas [23] established the identity

$$\int_0^1 \zeta(1-z,\{aq\})\,\zeta(1-z,\{bq\})dq = 2\Gamma^2(z)\frac{\zeta(2z)}{(2\pi)^{2z}}\left(\frac{(a,b)}{[a,b]}\right)^z \tag{1.13}$$

for $a,\ b \in \mathbb{N}$. Here (a,b) is the greatest common divisor of a and b and $[a,b]$ is their least common multiple.

The Hurwitz zeta function also plays a role in the evaluation of functional determinants that appear in mathematical physics. See [14] for a miscellaneous list of physical examples. The Hurwitz zeta function has also recently appeared in connection with the problem of a gas of non-interacting electrons in the background of a uniform magnetic field [13]. For instance, it is shown there that the *density of states* $g(E)$, in terms of which all thermodynamic functions are to be computed, can be written as

$$g(E) = V\frac{4\pi}{h^3}(2e\hbar B)^{1/2}E\,\mathcal{H}_{1/2}\left(\frac{E^2-m^2}{2e\hbar B}\right), \tag{1.14}$$

where V stands for volume, B for magnetic field, m is the electron mass, and

$$\mathcal{H}_z(q) := \zeta(z,\{q\}) - \zeta(z,q+1) - \frac{1}{2}q^{-z}. \tag{1.15}$$

As before, $\{q\}$ in (1.15) denotes the fractional part of q.

General information about $\zeta(z,q)$ appears in [7] and [30].

In this paper we derive a series of formulae for definite integrals containing $\zeta(z,q)$ in the integrand. A search of the standard tables of integrals reveals very few examples in [25] and none in [16]. For instance, in [25], Section 1.2.1 we find the indefinite integral

$$\int \zeta(z,q)\,dq = \frac{1}{1-z}\zeta(z-1,q), \tag{1.16}$$

which is an elementary consequence of

$$\frac{\partial}{\partial q}\zeta(z-1,q) = (1-z)\zeta(z,q). \tag{1.17}$$

Section 2.3.1 of [25] gives two definite integrals:

$$\int_0^\infty q^{\alpha-1}\zeta(z,a+bq)\,dq = b^{-\alpha}B(\alpha,z-\alpha)\zeta(z-\alpha,a)$$

for $a, b \in \mathbb{R}^+$, $0 < \text{Re}(\alpha) < \text{Re}(z) - 1$; and

$$\int_0^\infty q^{\alpha-1}[\zeta(z,q) - q^{-z}]dq = B(\alpha, z-\alpha)\zeta(z-\alpha)$$

for $0 < \text{Re}(\alpha) < \text{Re}(z) - 1$, where $B(x, y)$ is the beta function. The second integral is actually a special case of the first with $a = b = 1$. The only other example in [25] is the evaluation of one of the Fourier coefficients of $\zeta(z, q)$ in Section 2.3.1:

$$\int_0^1 \sin(2\pi q)\zeta(z,q)\,dq = \frac{(2\pi)^z}{4\Gamma(z)} \csc\left(\frac{z\pi}{2}\right) \tag{1.18}$$

for $1 < \text{Re}(z) < 2$.

The tables [16, 25] do contain many examples involving the special case

$$\zeta(1-m, q) = -\frac{1}{m} B_m(q) \tag{1.19}$$

for $m \in \mathbb{N}$, $q \in \mathbb{R}_0^+$, where $B_m(q)$ are the Bernoulli polynomials defined by their generating function

$$\frac{te^{qt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(q) \frac{t^m}{m!} \tag{1.20}$$

for $|t| < 2\pi$. These polynomials can be expressed as

$$B_m(q) = \sum_{k=0}^{m} \binom{m}{k} B_k q^{m-k} \tag{1.21}$$

in terms of the Bernoulli numbers $B_m = B_m(0)$. The latter are rational numbers; for example, $B_0 = 1$, $B_1 = -1/2$, and, $B_2 = 1/6$. The Bernoulli numbers of odd index B_{2m+1} vanish for $m \geq 1$, and those with even index satisfy $(-1)^{m+1} B_{2m} > 0$.

The relation (1.21) can be inverted to produce

$$q^n = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} B_j(q), \tag{1.22}$$

and since $B_j(1-q) = (-1)^j B_j(q)$, we also have

$$(1-q)^n = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} B_j(q). \tag{1.23}$$

For example,

$$B_0(q) = 1, \quad B_1(q) = q - \frac{1}{2}, \quad B_2(q) = q^2 - q + \frac{1}{6}$$

yield

$$1 = B_0(q), \quad q = B_1(q) + \frac{1}{2}B_0(q), \quad q^2 = B_2(q) + B_1(q) + \frac{1}{3}B_0(q).$$

The results presented here are consequences of the Fourier expansion of $\zeta(z, q)$:

$$\zeta(z, q) = \frac{2\Gamma(1 - z)}{(2\pi)^{1-z}} \times \left(\sin\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi qn)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi qn)}{n^{1-z}}\right). \tag{1.24}$$

This expansion, valid for $\text{Re}(z) < 0$ and $0 < q < 1$, is due to Hurwitz and is derived in [31], page 268. A proof of (1.24) based upon the representation[1]

$$\zeta(z, q) = z\int_{-q}^{\infty}\frac{\lfloor x\rfloor - x + \frac{1}{2}}{(x + q)^{z+1}}\,dx \tag{1.25}$$

appears in [8]. The result

$$\int_0^1 \zeta(z, q)\,dq = 0, \tag{1.26}$$

valid for $\text{Re}(z) < 0$, follows directly from the representation (1.24). Although the Fourier expansion is derived strictly for $\text{Re}(z) < 0$, it also holds for the boundary value $z = 0$. We shall thus simply take $z \in \mathbb{R}_0^- := (-\infty, 0]$ in most of the formulae presented below.

Our goal is to employ the representation (1.24) to evaluate definite integrals containing $\zeta(z, q)$ in the integrand. These evaluations can be seen as examples of the *Hurwitz transform* defined by

$$\mathfrak{H}(f) := \int_0^1 f(q)\zeta(z, q)\,dq. \tag{1.27}$$

Properties of $\mathfrak{H}$ and its uses will be discussed elsewhere. The relation (1.19) between Bernoulli polynomials and the Hurwitz zeta function yields, for each evaluation of the Hurwitz transform, an explicit formula for an integral of the type

$$\mathfrak{B}_m(f) := \int_0^1 f(q)B_m(q)\,dq, \tag{1.28}$$

and by (1.22) the evaluation of the moments of the function f

$$\mathfrak{M}_n(f) := \int_0^1 q^n f(q)\,dq. \tag{1.29}$$

We have attempted to evaluate symbolically, using Mathematica 4.0 and/or Maple V, each of the examples presented here. The few cases in which this attempt was successful are so indicated.

The relations

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{2(2n)!}, \quad n \in \mathbb{N}_0, \tag{1.30}$$

$$\zeta(1-n) = \frac{(-1)^{n+1}B_n}{n}, \quad n \in \mathbb{N}, \tag{1.31}$$

$$\zeta'(-2n) = (-1)^n \frac{(2n)!\,\zeta(2n+1)}{2\,(2\pi)^{2n}}, \quad n \in \mathbb{N}, \tag{1.32}$$

$$\zeta'(0) = -\ln\sqrt{2\pi}, \tag{1.33}$$

and Riemann's functional equation

$$\zeta(1-s) = \frac{\zeta(s)(2\pi)^{1-s}}{2\Gamma(1-s)\sin(\pi s/2)} \tag{1.34}$$

$$= 2\cos\left(\frac{\pi s}{2}\right)\frac{\zeta(s)\Gamma(s)}{(2\pi)^s} \tag{1.35}$$

will be used to simplify the integrals discussed below. The form (1.35) follows from (1.34) by use of the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x} \tag{1.36}$$

for the gamma function. The basic relation between the beta and gamma functions,

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \tag{1.37}$$

will also be employed throughout.

2. The Fourier expansion of $\zeta(z, q)$

In this section we employ the Fourier expansion (1.24) for $\zeta(z, q)$ to evaluate definite integrals of the form

$$\mathfrak{H}(f) := \int_0^1 f(q)\zeta(z,q)\,dq. \tag{2.1}$$

The expansion is valid for $z \leq 0$.

We first record the Fourier coefficients of $\zeta(z, q)$. These can be read directly from (1.24).

Proposition 2.1. *The Fourier coefficients of $\zeta(z, q)$ are given by*

$$\int_0^1 \sin(2k\pi q)\zeta(z,q)\,dq = \frac{(2\pi)^z\,k^{z-1}}{4\Gamma(z)}\csc\left(\frac{z\pi}{2}\right) \tag{2.2}$$

and

$$\int_0^1 \cos(2k\pi q)\zeta(z,q)\,dq = \frac{(2\pi)^z k^{z-1}}{4\Gamma(z)}\sec\left(\frac{z\pi}{2}\right). \tag{2.3}$$

Proof: The orthogonality of the trigonometric functions and (1.24) yield

$$\int_0^1 \sin(2k\pi q)\zeta(z,q)\,dq = \frac{\Gamma(1-z)}{(2\pi k)^{1-z}}\cos\left(\frac{\pi z}{2}\right). \tag{2.4}$$

Now use the reflection formula (1.36) to obtain (2.2). The calculation of (2.3) is similar. □

The theorem below reduces the evaluation of an integral of the type considered here to the evaluation of a Dirichlet series formed with the Fourier coefficients of the integrand. The remainder of the paper are applications of this result.

Theorem 2.2. *Let $f(w,q)$ be defined for $q \in [0,1]$ and a parameter w. Let*

$$f(w,q) = a_0(w) + \sum_{n=1}^{\infty} a_n(w)\cos(2\pi qn) + b_n(w)\sin(2\pi qn) \tag{2.5}$$

be its Fourier expansion, so that

$$a_n(w) = 2\int_0^1 f(w,q)\cos(2\pi qn)\,dq, \quad n \geq 0, \tag{2.6}$$

$$b_n(w) = 2\int_0^1 f(w,q)\sin(2\pi qn)\,dq, \quad n \geq 1. \tag{2.7}$$

Then, for $z \in \mathbb{R}_0^-$,

$$\int_0^1 f(w,q)\zeta(z,q)\,dq = \frac{\Gamma(1-z)}{(2\pi)^{1-z}}\left(\sin\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{a_n(w)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{b_n(w)}{n^{1-z}}\right) \tag{2.8}$$

and

$$\int_0^1 f(w,q)\zeta(z,1-q)\,dq = \frac{\Gamma(1-z)}{(2\pi)^{1-z}}\left(\sin\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{a_n(w)}{n^{1-z}} - \cos\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{b_n(w)}{n^{1-z}}\right). \tag{2.9}$$

Proof: Multiply (2.5) by $\zeta(z,q)$, integrate over [0, 1], and apply (2.2) and (2.3) to give (2.8). Observe that the integral of $\zeta(z,q)$ over [0, 1] vanishes, so there is no contribution from $a_0(w)$. The second result follows from the fact that the Fourier expansion of $\zeta(z,1-q)$ differs from that of $\zeta(z,q)$ given in (1.24) only in the sign of the last term. □

3. Product of two zeta and related functions

In this section we evaluate integrals with integrands consisting of products of two Hurwitz zeta functions. Classical relations for the Bernoulli polynomials are obtained as corollaries.

Theorem 3.1. *Let $z,\ z' \in \mathbb{R}_0^-$. Then*

$$\int_0^1 \zeta(z',q)\zeta(z,q)\,dq = \frac{2\Gamma(1-z)\Gamma(1-z')}{(2\pi)^{2-z-z'}}\zeta(2-z-z')\cos\left(\frac{\pi(z-z')}{2}\right) \quad (3.1)$$

$$= -\zeta(z+z'-1)B(1-z,1-z')\,\frac{\cos(\pi(z-z')/2)}{\cos(\pi(z+z')/2)}. \quad (3.2)$$

Similarly,

$$\int_0^1 \zeta(z',q)\zeta(z,1-q)\,dq = -\frac{2\Gamma(1-z)\Gamma(1-z')}{(2\pi)^{2-z-z'}}\zeta(2-z-z')\cos\left(\frac{\pi(z+z')}{2}\right) \quad (3.3)$$

$$= \zeta(z+z'-1)B(1-z,1-z'). \quad (3.4)$$

Proof: The expansion (1.24) shows that the coefficients of $\zeta(z',q)$ are given by

$$a_n = \frac{2\Gamma(1-z')\sin(\pi z'/2)}{(2\pi)^{1-z'}}\frac{1}{n^{1-z'}},$$

$$b_n = \frac{2\Gamma(1-z')\cos(\pi z'/2)}{(2\pi)^{1-z'}}\frac{1}{n^{1-z'}}.$$

Theorem 2.2 then yields (3.1). Now use Riemann's relation (1.34) for the ζ-function to obtain (3.2). The proofs of (3.3) and (3.4) are similar. □

Note. An integral related to (3.1) appears in [4].

Example 3.2. Let $z \in \mathbb{R}_0^-$. Then

$$\int_0^1 \zeta^2(z,q)\,dq = 2\Gamma^2(1-z)(2\pi)^{2z-2}\zeta(2-2z) \quad (3.5)$$

and

$$\int_0^1 \zeta(z,q)\zeta(z,1-q)\,dq = -2\Gamma^2(1-z)(2\pi)^{2z-2}\zeta(2-2z)\cos(\pi z). \quad (3.6)$$

Proof: Let $z = z'$ in (3.1) and (3.3). □

Example 3.3. Let $m \in \mathbb{N}_0$. Then

$$\int_0^1 B_m^2(q)\,dq = \frac{|B_{2m}|}{\binom{2m}{m}}. \quad (3.7)$$

Proof: For $m \geq 1$ let $z = 1 - m$ in (3.5). The case $m = 0$ is direct. □

Example 3.4. Let $m \in \mathbb{N}$. Then

$$\int_0^1 \zeta^2\left(-m+\frac{1}{2}, q\right) dq = \left(\frac{(2m)!}{2^{2m}\, m!}\right)^2 \frac{\zeta(2m+1)}{(2\pi)^{2m}}. \tag{3.8}$$

Proof: Let $z = -m + \frac{1}{2}$ in (3.5) and use

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2m)!}{2^{2m}\, m!}.$$

□

In particular, for $z = -\frac{1}{2}$ $(m = 1)$ we obtain

$$\int_0^1 \zeta^2\left(-\frac{1}{2}, q\right) dq = \frac{\zeta(3)}{16\pi^2}. \tag{3.9}$$

The next two examples present special cases of (3.2) that involve integrals of Bernoulli polynomials.

Example 3.5. Let $z \in \mathbb{R}_0^-$ and $m \in \mathbb{N}$. Then

$$\int_0^1 B_m(q)\zeta(z, q)\, dq = (-1)^{m+1} \frac{m!\, \zeta(z-m)}{(1-z)_m}, \tag{3.10}$$

where $(z)_k := z(z+1)(z+2)\ldots(z+k-1)$ is the Pochhammer symbol.

Proof: Let $z' = 1 - m$ in (3.2) to produce

$$\int_0^1 B_m(q)\zeta(z, q)\, dq = (-1)^{m+1}\, m\, B(1-z, m)\zeta(z-m). \tag{3.11}$$

The result then follows from $B(1-z, m) = (m-1)!/(1-z)_m$. □

The next formula appears as 2.4.2.2 in [25].

Example 3.6. Let $n, m \in \mathbb{N}$. Then

$$\int_0^1 B_m(q)B_n(q)\, dq = (-1)^{m+1} \frac{B_{m+n}}{\binom{m+n}{m}}, \tag{3.12}$$

The case $m = n$ confirms (3.7).

Proof: Let $z = 1 - n \in -\mathbb{N}_0$ in (3.10) to obtain

$$\begin{aligned}\int_0^1 B_m(q)B_n(q)\,dq &= \frac{(-1)^m n\, m!\,\zeta(1-n-m)}{(n)_m}\\ &= \frac{(-1)^m m!n!\zeta(n+m)}{(2\pi)^{n+m}}\,2\cos\left(\frac{\pi(m+n)}{2}\right)\end{aligned}$$

using (1.35). Both sides of (3.12) vanish for $n+m$ odd, and for $n+m$ even the result follows from (1.30). □

We now establish a formula for the moments of $\zeta(z, q)$.

Theorem 3.7. *The moments of the Hurwitz zeta function are given by*

$$\begin{aligned}\int_0^1 q^n\zeta(z,q)\,dq &= -n!\sum_{j=1}^{n}\frac{\zeta(z-j)}{(z-j)_j\,(n-j+1)!}\\ &= n!\sum_{j=1}^{n}(-1)^{j+1}\frac{\zeta(z-j)}{(1-z)_j\,(n-j+1)!}.\end{aligned} \tag{3.13}$$

Proof: We prove (3.13) by induction. The case $n = 1$ follows from (3.10) and the vanishing of the integral of $\zeta(z, q)$. For $n > 1$, integration by parts yields

$$\begin{aligned}\int_0^1 q^{n+1}\zeta(z,q)\,dq &= \frac{1}{1-z}\int_0^1 q^{n+1}\frac{\partial}{\partial q}\zeta(z-1,q)\,dq\\ &= \frac{\zeta(z-1)}{1-z} + \frac{(n+1)!}{1-z}\sum_{k=2}^{n+1}\frac{\zeta(z-k)}{(z-k)_{k-1}\,(n-k+2)!},\end{aligned}$$

where we have used (3.13) for power n. The final form is obtained from the identity $(1-z)\times(z-k)_{k-1} = -(z-k)_k$. □

Note. Analytic continuation extends (3.13) to $n - z + 1 > 0$. See Section 12.

A direct proof of (3.13) can be given using the expansion of q^n in terms of Bernoulli polynomials given in (1.22) and the evaluation (3.10):

$$\begin{aligned}\int_0^1 q^n\zeta(z,q)\,dq &= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}\int_0^1 B_j(q)\zeta(z,q)\,dq\\ &= \frac{1}{n+1}\sum_{j=1}^{n}\binom{n+1}{j}(-1)^{j+1}\frac{j!\,\zeta(z-j)}{(1-z)_j}.\end{aligned}$$

Noting the similitude between (1.22) and (1.23), the proof above can be imitated to give

Example 3.8. For $n \in \mathbb{N}$,

$$\int_0^1 (1-q)^n \zeta(z,q)\,dq = -n! \sum_{j=1}^{n} \frac{\zeta(z-j)}{(1-z)_j\,(n-j+1)!}. \tag{3.14}$$

The special case $z \in -\mathbb{N}_0$ in Theorem 3.7 yields the moments of the Bernoulli polynomials.

Example 3.9. Let $n, m \in \mathbb{N}$. Then

$$\begin{aligned} \int_0^1 q^n B_m(q)\,dq &= \frac{1}{n+1} \sum_{j=1}^{n} (-1)^{j+1} \frac{\binom{n+1}{j}}{\binom{m+j}{j}}\, B_{m+j} \\ &= \frac{n!\,m!}{(n+1+m)!} \sum_{j=1}^{n} (-1)^{j+1} \binom{n+1+m}{n+1-j} B_{m+j}. \end{aligned} \tag{3.15}$$

Proof: Apply (1.19) to write

$$\begin{aligned} \int_0^1 q^n B_m(q)\,dq &= -m \int_0^1 q^n \zeta(1-m,q)\,dq \\ &= m \sum_{j=1}^{n} (-1)^j \frac{\zeta(1-m-j)(j-1)!}{(m)_j} \binom{n}{j-1} \\ &= (-1)^{n+1} \sum_{j=1}^{n} (-1)^{j+1} \frac{B_{m+j}}{\frac{m+j}{m}\frac{(m)_j}{j!}} \binom{n+1}{j}, \end{aligned}$$

using (1.30) to go from the second to the third line. The final form follows from the identity

$$\frac{m+j}{m} \times \frac{(m)_j}{j!} = \binom{m+j}{j}. \qquad \square$$

4. The exponential function

In this section we evaluate the Hurwitz transform of the exponential function. The result is expressed in terms of the transcendental function

$$F(x,z) := \sum_{n=0}^{\infty} \zeta(n+2-z)x^n, \quad \text{for } |x| < 1. \tag{4.1}$$

Example 4.1. Let $z \in \mathbb{R}_0^-$ and $|t| < 1$. Then

$$\int_0^1 e^{2\pi tq} \zeta(z,q)\,dq = 2(1-e^{2\pi t}) \frac{\Gamma(1-z)}{(2\pi)^{2-z}} \times \mathrm{Re}[e^{\pi iz/2} F(it,z)], \tag{4.2}$$

where $F(x,z)$ is given in (4.1).

Proof: The generating function for the Bernoulli polynomials (1.20) yields

$$e^{qt} = \frac{e^t - 1}{t} \sum_{n=0}^{\infty} B_n(q) \frac{t^n}{n!},$$

so that

$$\int_0^1 e^{qt} \zeta(z, q)\, dq = \frac{e^t - 1}{t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 B_n(q) \zeta(z, q)\, dq.$$

Since $B_0(q) = 1$ and $\zeta(z, q)$ integrates to 0, the above sum effectively starts at $n = 1$. Thus (3.11) gives

$$\int_0^1 e^{qt} \zeta(z, q)\, dq = (e^t - 1) \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} B(1 - z, n + 1) \zeta(z - n - 1),$$

which can be written as

$$\int_0^1 e^{qt} \zeta(z, q)\, dq = \frac{2(e^t - 1)\Gamma(1 - z)}{(2\pi)^{2-z}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{t}{2\pi}\right)^n \zeta(n + 2 - z) \cos\left(\frac{\pi(z - n)}{2}\right)$$

using (1.35) and (1.37). Now replace t by $2\pi t$ and use the evaluation $\cos(\pi(z - n)/2) = \text{Re}(e^{i\pi(z-n)/2}) = \text{Re}((-i)^n e^{i\pi z/2})$ to yield the final result. □

The next example results from $z \in -\mathbb{N}_0$ in (4.2). It appears in [25]: 2.4.1.4.[2]

Example 4.2. Let $m \in \mathbb{N}$ and $|t| < 1$. Then

$$\int_0^1 e^{2\pi t q} B_m(q)\, dq = \frac{(-1)^m (e^{2\pi t} - 1)\, m!}{(2\pi t)^{m+1}} \times \left[1 - \pi t \coth(\pi t) - 2 \sum_{r=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \zeta(2r) t^{2r} \right]. \tag{4.3}$$

Proof: We discuss the case $m = 2k + 1$; the case of m even is similar. Let $z = 1 - m = -2k$ in (4.2). Then

$$\begin{aligned} \text{Re}[e^{\pi i z/2} F(it, z)] &= (-1)^k \,\text{Re}\,[F(it, -2k)] \\ &= (-1)^k \sum_{r=0}^{\infty} \zeta(2r + 2 + 2k)(-1)^r t^{2r} \\ &= -t^{2k+2} \left[\frac{1}{2} - \frac{\pi t}{2} \coth \pi t - \sum_{r=1}^{k} (-1)^r \zeta(2r) t^{2r} \right], \end{aligned}$$

where we have employed the identity

$$\coth \pi x = \frac{1}{\pi x} - \frac{2}{\pi x} \sum_{r=1}^{\infty} (-1)^r \zeta(2r) x^{2r} \tag{4.4}$$

that appears in [27], 3:14:5. □

5. The logsine function

This section contains examples involving the function $\ln(\sin \pi q)$. The standard tables [16] and [25] contain very few examples of this type. See Sections 4.224 and 4.322 in [16]. Some of the evaluations presented here are computable by Mathematica 4.0.

Example 5.1. Let $z \in \mathbb{R}_0^-$. Then

$$\int_0^1 \ln(\sin \pi q)\, \zeta(z, q)\, dq = -\frac{\Gamma(1-z)}{(2\pi)^{1-z}} \sin\left(\frac{\pi z}{2}\right) \zeta(2-z) \tag{5.1}$$

$$= -\frac{\zeta(z)\, \zeta(2-z)}{2\zeta(1-z)}, \tag{5.2}$$

where the second result follows from (5.1) when $z \neq 0$ by use of (1.34).

Proof: The Fourier coefficients of $\ln(\sin \pi q)$ are

$$\int_0^1 \ln(\sin \pi q) \sin(2n\pi q)\, dq = 0$$

and

$$\int_0^1 \ln(\sin \pi q) \cos(2n\pi q)\, dq = \begin{cases} -\ln 2 & \text{if } n = 0, \\ -\dfrac{1}{2n} & \text{if } n > 0. \end{cases}$$

These appear in [16] 4.384. Thus (5.1) follows from Theorem 2.2. □

Example 5.2. Let $m \in \mathbb{N}$. Then

$$\int_0^1 \ln(\sin \pi q)\, B_m(q)\, dq = \begin{cases} (-1)^{m/2}(2\pi)^{-m} m! \zeta(m+1) & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases} \tag{5.3}$$

Proof: Let $z = 1 - m \in -\mathbb{N}_0$ in (5.2) giving

$$\int_0^1 \ln(\sin \pi q)\, B_m(q)\, dq = \frac{m\zeta(1-m)\zeta(1+m)}{2\zeta(m)}. \tag{5.4}$$

Now use (1.35) to obtain the result. □

The next example evaluates the moments of $\ln(\sin \pi q)$.

Example 5.3. Let $n \in \mathbb{N}_0$. Then

$$\int_0^1 q^n \ln(\sin \pi q)\, dq = -\frac{\ln 2}{n+1} + n! \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \zeta(2k+1)}{(2\pi)^{2k}\, (n+1-2k)!}. \tag{5.5}$$

Proof: Using (1.22) we have

$$\int_0^1 q^n \ln(\sin \pi q)\, dq = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \int_0^1 \ln(\sin \pi q)\, B_j(q)\, dq.$$

The result now follows by (5.3) and the classical value

$$\int_0^1 \ln(\sin \pi q)\, dq = -\ln 2. \tag{5.6}$$

An elementary evaluation of (5.6) appears in [6].

The first few cases are

$$\begin{aligned}
\int_0^1 q \, \ln(\sin \pi q)\, dq &= -\frac{1}{2} \ln 2, \\
\int_0^1 q^2 \, \ln(\sin \pi q)\, dq &= -\frac{1}{3} \ln 2 - \frac{\zeta(3)}{2\pi^2}, \\
\int_0^1 q^3 \, \ln(\sin \pi q)\, dq &= -\frac{1}{4} \ln 2 - \frac{3\zeta(3)}{4\pi^2}, \\
\int_0^1 q^4 \ln(\sin \pi q)\, dq &= -\frac{1}{5} \ln 2 - \frac{\zeta(3)}{\pi^2} + \frac{3\zeta(5)}{2\pi^4}.
\end{aligned} \tag{5.7}$$

These evaluations can be confirmed by Mathematica 4.0. □

6. The loggamma function

This section contains evaluations involving the function $\ln \Gamma(q)$. None of the examples presented here were computable by a symbolic language.

Example 6.1. Let $z \in \mathbb{R}_0^-$. Then

$$\begin{aligned}
\int_0^1 \ln \Gamma(q)\, \zeta(z, q)\, dq = {} & \frac{\Gamma(1-z)}{(2\pi)^{2-z}} \zeta(2-z) \\
& \times \left[\pi \sin\left(\frac{\pi z}{2}\right) + 2 \cos\left(\frac{\pi z}{2}\right) \left\{ A - \frac{\zeta'(2-z)}{\zeta(2-z)} \right\} \right],
\end{aligned} \tag{6.1}$$

where

$$A := 2\ln\sqrt{2\pi} + \gamma = -2\frac{d}{dz}\left(\zeta(z)\Gamma(1-z)\right)\Big|_{z=0} \tag{6.2}$$

and γ is Euler's constant.

Proof: The Fourier coefficients of $\ln\Gamma(q)$ appear in [16] 6.443.1 and 6.443.3 as

$$\int_0^1 \ln\Gamma(q)\,\sin(2\pi nq)\,dq = \frac{A+\ln n}{2\pi n}, \quad n \in \mathbb{N}, \tag{6.3}$$

$$\int_0^1 \ln\Gamma(q)\,\cos(2\pi nq)\,dq = \frac{1}{4n}, \quad n \in \mathbb{N}. \tag{6.4}$$

Thus

$$a_n = \frac{1}{2n} \quad \text{and} \quad b_n = \frac{A+\ln n}{\pi n},$$

where A is defined in (6.2). The evaluations

$$\sum_{n=1}^{\infty} \frac{1}{2n^{2-z}} = \frac{1}{2}\zeta(2-z) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{A+\ln n}{n^{2-z}} = A\zeta(2-z) - \zeta'(2-z)$$

yield (6.1). □

Note. The integral

$$\int_0^1 \ln\Gamma(q)\,\cos((2n+1)\pi q)\,dq = \frac{2}{\pi^2}\left(\frac{\gamma + 2\ln\sqrt{2\pi}}{(2n+1)^2} + 2\sum_{k=2}^{\infty} \frac{\ln k}{4k^2-(2n+1)^2}\right),$$

a companion to (6.4), was evaluated by Kölbig in [19]. This was recorded as 0 as late as in the fourth edition of [16]. The fifth edition contains the correct value.

Example 6.2. Let $m \in \mathbb{N}$. Then

$$\int_0^1 B_{2m}(q)\ln\Gamma(q)\,dq = (-1)^{m+1}\frac{(2m)!\zeta(2m+1)}{2(2\pi)^{2m}} = -\zeta'(-2m), \tag{6.5}$$

$$\int_0^1 B_{2m-1}(q)\ln\Gamma(q)\,dq = \frac{B_{2m}}{2m} \times \left[\frac{\zeta'(2m)}{\zeta(2m)} - A\right]. \tag{6.6}$$

Proof: Replace in (6.1) the variable z by $1-2m$ and $2-2m$ respectively. Then use (1.32) in the first case and (1.30) in the second. □

An alternative approach. The evaluation in Example 6.2 can also be obtained by integrating

$$\frac{d}{dz}\zeta(z,q)\Big|_{z=0} = \ln\Gamma(q) - \ln\sqrt{2\pi} \tag{6.7}$$

to produce

$$\int_0^1 B_m(q) \ln \Gamma(q)\, dq = \ln \sqrt{2\pi} \int_0^1 B_m(q)\, dq + \left.\frac{d}{dz}\right|_{z=0} \int_0^1 B_m(q) \zeta(z, q)\, dq. \tag{6.8}$$

The relation (6.7) can be found in [16] 9.533.3. To evaluate (6.8) differentiate (3.10) to produce

$$\int_0^1 B_m(q) \ln \Gamma(q)\, dq = \ln \sqrt{2\pi}\, \delta_{m,0} + (1 - \delta_{m,0})(-1)^{m+1}[H_m \zeta(-m) + \zeta'(-m)].$$

Here $\delta_{m,0}$ is Kronecker's delta and $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ is the m-th harmonic number. Use has been made of the result

$$\left.\frac{d}{dz}(1-z)_k\right|_{z=0} = -k!H_k. \tag{6.9}$$

According to the parity of m we have

$$\int_0^1 B_m(q) \ln \Gamma(q)\, dq = \begin{cases} -\zeta'(-m) & m = 0, 2, 4, \ldots, \\ H_m \zeta(-m) + \zeta'(-m) & m = 1, 3, \ldots \end{cases} \tag{6.10}$$

(for $m = 0$ we have used (1.33)). The result (6.10) for odd m is seen to be equivalent to (6.6) after use of the identity

$$\frac{\zeta'(1-2k)}{\zeta(1-2k)} + \frac{\zeta'(2k)}{\zeta(2k)} = \ln 2\pi + \gamma - H_{2k-1}, \quad k \in \mathbb{N}, \tag{6.11}$$

which can be derived by differentiating Riemann's relation (1.34) and evaluating at $s = 2k$.

Example 6.3. The case $m = 1$ in (6.6) yields, using $\zeta(2) = \pi^2/6$,

$$\int_0^1 \left(q - \frac{1}{2}\right) \ln \Gamma(q)\, dq = \frac{1}{12}\left(\frac{6\zeta'(2)}{\pi^2} - 2 \ln \sqrt{2\pi} - \gamma\right). \tag{6.12}$$

The case $m = 1$ in (6.5) gives

$$\int_0^1 \left(q^2 - q + \frac{1}{6}\right) \ln \Gamma(q)\, dq = \frac{\zeta(3)}{4\pi^2}. \tag{6.13}$$

Example 6.4. Let $n \in \mathbb{N}$. Then

$$\int_0^1 q^n \ln \Gamma(q)\, dq = \frac{1}{n+1} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n+1}{2k-1} \frac{(2k)!}{k(2\pi)^{2k}} [A\zeta(2k) - \zeta'(2k)] \\ - \frac{1}{n+1} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+1}{2k} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) + \frac{\ln \sqrt{2\pi}}{n+1}. \quad (6.14)$$

Proof: Use the expression (1.22) to write

$$\int_0^1 q^n \ln \Gamma(q)\, dq = \frac{1}{n+1} \sum_{j=1}^{n} \binom{n+1}{j} \int_0^1 \ln \Gamma(q) B_j(q)\, dq \\ + \frac{1}{n+1} \int_0^1 \ln \Gamma(q)\, dq.$$

The value

$$\int_0^1 \ln \Gamma(q)\, dq = \ln \sqrt{2\pi} \quad (6.15)$$

is then obtained from (6.7) and (1.26). The result now follows from (6.5) and (6.6). □

The formula (6.14) yields

$$\int_0^1 q \ln \Gamma(q)\, dq = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{3} \ln \sqrt{2\pi} - \frac{\gamma}{12},$$

$$\int_0^1 q^2 \ln \Gamma(q)\, dq = \frac{\zeta'(2)}{2\pi^2} + \frac{\zeta(3)}{4\pi^2} + \frac{1}{6} \ln \sqrt{2\pi} - \frac{\gamma}{12},$$

$$\int_0^1 q^3 \ln \Gamma(q)\, dq = \frac{\zeta'(2)}{2\pi^2} + \frac{3\zeta(3)}{8\pi^2} - \frac{3\zeta'(4)}{4\pi^4} + \frac{1}{10} \ln \sqrt{2\pi} - \frac{3\gamma}{40}.$$

None of these examples could be evaluated symbolically.

Note. Gosper [15] presents a series of interesting evaluations of definite integrals of $\ln \Gamma(q)$. For example

$$\int_0^{1/2} \ln \Gamma(q+1)\, dq = \frac{\gamma}{8} + \frac{3 \ln \sqrt{2\pi}}{4} - \frac{13 \ln 2}{24} - \frac{3\zeta'(2)}{4\pi^2} - \frac{1}{2} \quad (6.16)$$

and

$$\int_0^{1/4} \ln \Gamma(q+1)\, dq = \frac{3\gamma}{32} + \frac{7 \ln \sqrt{2\pi}}{16} - \frac{\ln 2}{2} - \frac{9\zeta'(2)}{16\pi^2} + \frac{G}{4\pi} - \frac{1}{4}, \quad (6.17)$$

where

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \tag{6.18}$$

is Catalan's constant.

Note. The results discussed here and those of Gosper's referred to in the previous note are special cases of the indefinite integral

$$\int q^n \ln\Gamma(q)\,dq = -\zeta'(0)\frac{q^{n+1}}{n+1} + n!\sum_{k=1}^{n+1}(-1)^{k+1}\frac{q^{n+1-k}}{k!(n+1-k)!}\left[\zeta_z(-k,q) - \frac{H_k}{k+1}B_{k+1}(q)\right],$$

where H_k is the k-th harmonic number and

$$\zeta_z(-k,q) := \left.\frac{\partial}{\partial z}\right|_{z=-k} \zeta(z,q). \tag{6.19}$$

These results can be expressed in terms Gosper's *negapolygammas* $\psi_{-k}(q)$ [15] in view of the relation

$$\zeta_z(-k,q) = \frac{H_k}{k+1}B_{k+1}(q) + q^k\zeta'(0) + k!\,\psi_{-k}(q),$$

where

$$\begin{aligned}\psi_{-1}(q) &= \ln\Gamma(q),\\ \psi_{-k}(q) &= \int \psi_{-k+1}(q)\,dq, \quad k \geq 2.\end{aligned}$$

Details will appear in [10].

7. Differentiation results

In this section we discuss evaluation of certain integrals that appear from (3.1) after differentiation with respect to the parameters z and z'. The special values $z = 0$ and $z' = 0$ produce evaluations containing the loggamma function, in view of (6.7). In particular, as was pointed out earlier, the result

$$\int_0^1 \ln\Gamma(q)\,dq = \ln\sqrt{2\pi} \tag{7.1}$$

follows directly from (6.7). The integrals considered here complement those considered in Section 6.

Proposition 7.1. *For $z, z' \in \mathbb{R}_0^-$ we have*

$$\int_0^1 \frac{d}{dz}\zeta(z,q)\zeta(z',q)\,dq = -\frac{2\Gamma(1-z)\Gamma(1-z')}{(2\pi)^{2-z-z'}}\zeta(2-z-z')\,\cos\omega \times\left[\frac{\zeta'(2-z-z')}{\zeta(2-z-z')} + \frac{\pi}{2}\tan\omega - 2\ln\sqrt{2\pi} + \psi(1-z)\right], \tag{7.2}$$

where $\omega = \pi(z-z')/2$ and $\psi(z)$ is the digamma function defined in (1.8).

Proof: Direct differentiation of (3.1). □

In particular, for $z = z' = 0$ we obtain (6.12).

Example 7.2. Differentiating (3.1) with respect to z and then z', evaluating at $z = z' = 0$, and using (6.7) yields

$$\int_0^1 (\ln\Gamma(q))^2\,dq = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{1}{3}\gamma\ln\sqrt{2\pi} + \frac{4}{3}\left(\ln\sqrt{2\pi}\right)^2 - (\gamma + 2\ln\sqrt{2\pi})\frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2}. \tag{7.3}$$

Example 7.3. Differentiating (5.1) with respect to z and then setting $z = 0$ yields, after using (5.6),

$$\int_0^1 \ln\sin\pi q\,\ln\Gamma(q)\,dq = -\ln 2\ln\sqrt{2\pi} - \frac{\pi^2}{24}. \tag{7.4}$$

8. An expression for Catalan's constant

In his discussion of Entry 17(v) of Chapter 8 of Ramanujan's Notebooks, Berndt [9] page 200, introduces the function

$$G(z,q) := \zeta(z,q) - \zeta(z,1-q) \tag{8.1}$$

and gives its Fourier expansion

$$G(z,q) = 4\Gamma(1-z)\cos\left(\frac{\pi z}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2\pi kq)}{(2\pi k)^{1-z}}. \tag{8.2}$$

This is an immediate consequence of the Fourier expansion (1.24) for $\zeta(z,q)$.

In terms of $G(z,q)$ we can define an anti-symmetrized Hurwitz transform,

$$\mathfrak{H}_A(f) := \frac{1}{2}\int_0^1 f(w,q)G(z,q)\,dq. \tag{8.3}$$

It is straightforward to show that for a function $f(w, q)$ with Fourier expansion as in Theorem (2.2) one obtains

$$\frac{1}{2}\int_0^1 f(w,q)G(z,q)\,dq = \frac{\Gamma(1-z)\cos(\pi z/2)}{(2\pi)^{1-z}}\sum_{n=1}^{\infty}\frac{b_n(w)}{n^{1-z}}. \tag{8.4}$$

As a particular example we compute the anti-symmetrized Hurwitz transform of $\sec(\pi q)$ and obtain as a corollary an expression for Catalan's constant.

Example 8.1. The anti-symmetrized Hurwitz transform of $\sec(\pi q)$ is

$$\frac{1}{2}\int_0^1 \frac{\zeta(z,q)-\zeta(z,1-q)}{\cos(\pi q)}\,dq = \frac{16\Gamma(1-z)\cos(\pi z/2)}{(2\pi)^{2-z}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{1-z}}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}.$$

Proof: In [16] 3.612.5 we find

$$\int_0^1 \frac{\sin(2n\pi q)}{\cos(\pi q)}\,dq = (-1)^{n+1}\frac{4}{\pi}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}. \tag{8.5}$$

A straightforward application of (8.4) completes the proof. □

The special case $z = 0$ yields the following result.

Proposition 8.2. *The Catalan constant G, defined in* (8.18), *is given by*

$$G = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}. \tag{8.6}$$

Proof: Put $z = 0$ in Example 8.1 to obtain

$$\int_0^1 \frac{\frac{1}{2}-q}{\cos(\pi q)}\,dq = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}. \tag{8.7}$$

The change of variable $t = \pi(\frac{1}{2} - q)$ then produces

$$\int_0^1 \frac{\frac{1}{2}-q}{\cos(\pi q)}\,dq = \frac{2}{\pi^2}\int_0^{\pi/2}\frac{t}{\sin t}\,dt = \frac{4G}{\pi^2} \tag{8.8}$$

since the second integral equals $2G$. □

Note. The direct symbolic evaluation of the integral in (8.7) yields

$$\int_0^1 \frac{\frac{1}{2}-q}{\cos(\pi q)}\,dq = \frac{1}{16\pi}\left[\frac{32G}{\pi} - {}_4F_3\left(\begin{matrix}1\ 1\ \frac{3}{2}\ \frac{3}{2}\\ 2\ 2\ 2\end{matrix};1\right) + 16\ln 2\right],$$

and thus

$$G = \frac{\pi}{32}\left[16\ln 2 - {}_4F_3\left(\begin{matrix}1\ 1\ \frac{3}{2}\ \frac{3}{2}\\ 2\ 2\ 2\end{matrix};1\right)\right]. \tag{8.9}$$

This form of Catalan's constant appears in [25]: 7.5.3.120, and (8.8) is Entry 14 in the list of expressions for G compiled by Adamchik in [1], but (8.6) does not appear there, nor in the more complete compilation of Bradley [12].

Example 8.3. Let $m \in \mathbb{N}_0$. Then

$$\int_0^1 \sec(\pi q) B_{2m+1}(q)\, dq = (-1)^{m+1}\frac{16(2m+1)!}{(2\pi)^{2m+2}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2m+1}}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1} \tag{8.10}$$

$$= (-1)^m \frac{4(2m+1)!}{(2\pi)^{2m+2}} \times {\sum}^* \frac{\psi(n/2+1/4)}{n^{2m+1}}, \tag{8.11}$$

where the sum extends over $n \in \mathbb{Z},\ n \neq 0$.

Proof: The value $z = -2m$ in Example 8.1 yields (8.10). To prove (8.11) it is enough to establish the identity

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2m+1}}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1} = -\frac{1}{4}{\sum}^* \frac{\psi(n/2+1/4)}{n^{2m+1}}. \tag{8.12}$$

The internal sum in (8.12) can be written as

$$\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1} = \frac{\pi}{4} + \frac{1}{4}(-1)^n\left[\psi(n/2+1/4) - \psi(n/2+3/4)\right]. \tag{8.13}$$

Logarithmic differentiation of the reflection formula (1.36) for the gamma function yields

$$\psi(1-x) = \psi(x) + \pi \operatorname{cotg} \pi x,$$

so that, evaluating at $x = 1/4 - n/2$,

$$\psi(1/4+n/2) - \psi(3/4+n/2) = \psi(1/4+n/2) - \psi(1/4-n/2) - (-1)^n\pi.$$

Thus

$$\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1} = \frac{1}{4}(-1)^n\left[\psi(1/4+n/2) - \psi(1/4-n/2)\right] \tag{8.14}$$

and (8.12) is established. □

9. Clausen and related functions

In this section we evaluate the Hurwitz transform of the Clausen functions $\mathrm{Cl}_n(q)$. These functions are defined by

$$\mathrm{Cl}_{2n}(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n}}, \quad n \geq 1 \tag{9.1}$$

and

$$\mathrm{Cl}_{2n+1}(x) := \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n+1}}, \quad n \geq 0. \tag{9.2}$$

Extensive information about these functions can be found in [20], Chapter 4. For example,

$$\mathrm{Cl}_1(x) = -\ln|2\sin(x/2)|.$$

More generally, one can define the Clausen functions in terms of the polylogarithm on the unit circle as

$$\begin{aligned} \mathrm{Cl}_{2n}(x) &:= \ \mathrm{Im}\,\mathrm{Li}_{2n}(e^{ix}), \\ \mathrm{Cl}_{2n+1}(x) &:= \mathrm{Re}\,\mathrm{Li}_{2n+1}(e^{ix}), \end{aligned} \tag{9.3}$$

where, for $|z| \leq 1$,

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad n \in \mathbb{N}. \tag{9.4}$$

The Fourier expansion of $\mathrm{Li}_n(z)$ on the unit circle,

$$\mathrm{Li}_n(e^{2\pi qi}) = \sum_{k=1}^{\infty} \frac{\cos(2\pi kq)}{k^n} + i \sum_{k=1}^{\infty} \frac{\sin(2\pi kq)}{k^n}, \quad 0 \leq q < 1, \tag{9.5}$$

leads us, in view of Theorem 2.2, to the next example.

Example 9.1. Let $z \in \mathbb{R}_0^-$. Then

$$\int_0^1 \mathrm{Li}_n(e^{2\pi qi})\zeta(z,q)\,dq = \frac{\Gamma(1-z)}{(2\pi)^{1-z}}\, e^{i\frac{\pi}{2}(1-z)}\zeta(1-z+n). \tag{9.6}$$

As immediate consequences we have the next three examples.

Example 9.2. Let $z \in \mathbb{R}_0^-$. Then

$$\int_0^1 \mathrm{Cl}_{2n}(2\pi q)\zeta(z,q)\,dq = \frac{\Gamma(1-z)\cos(\pi z/2)}{(2\pi)^{1-z}}\zeta(1-z+2n) \tag{9.7}$$

and

$$\int_0^1 \mathrm{Cl}_{2n+1}(2\pi q)\zeta(z,q)\,dq = \frac{\Gamma(1-z)\sin(\pi z/2)}{(2\pi)^{1-z}}\zeta(2-z+2n). \tag{9.8}$$

Example 9.3. Let $m \in \mathbb{N}$. Then

$$\int_0^1 B_m(q)\,\mathrm{Cl}_{2n}(2\pi q)\,dq = \begin{cases} 0 & \text{if } m \text{ is even,} \\ (-1)^{\frac{m+1}{2}}\, m!(2\pi)^{-m}\,\zeta(m+2n) & \text{if } m \text{ is odd,} \end{cases}$$

and

$$\int_0^1 B_m(q)\,\mathrm{Cl}_{2n+1}(2\pi q)\,dq = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ (-1)^{\frac{m}{2}+1}\, m!(2\pi)^{-m}\,\zeta(m+2n+1) & \text{if } m \text{ is even.} \end{cases}$$

Example 9.4. Let $m \in \mathbb{N}$. Then

$$\int_0^1 q^m\,\mathrm{Cl}_{2n}(2\pi q)\,dq = m! \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^{j+1}\,\zeta(2n+2j+1)}{(m-2j)!\,(2\pi)^{2j+1}}$$

and

$$\int_0^1 q^m\,\mathrm{Cl}_{2n+1}(2\pi q)\,dq = m! \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^{j+1}\,\zeta(2n+2j+1)}{(m-2j+1)!\,(2\pi)^{2j}}.$$

10. Eisenstein series

In this section we compute the Hurwitz transform of functions related to the Eisenstein series $G_k(\tau)$.

The Eisenstein series defined by[3]

$$G_k(\tau) := \sum_{m,n}{}' \frac{1}{(m\tau+n)^{2k}}, \tag{10.1}$$

for $k \geq 2$ and $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$, are periodic functions with expansion ([26], page 92):

$$G_k(\tau) = 2\zeta(2k) + 2\frac{(-1)^k(2\pi)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n\tau}, \tag{10.2}$$

where

$$\sigma_s(n) := \sum_{d|n} d^s. \tag{10.3}$$

These series appear as coefficients in the cubic

$$y^2 = 4x^3 - 60G_2x - 140G_3 \tag{10.4}$$

that represents the torus $\mathbb{C}/\mathbb{L}$, with $\mathbb{L} = \mathbb{Z} \oplus \tau\mathbb{Z}$. See [22] for details.

Write $\tau = q + it$, with $t > 0$ and $0 \leq q \leq 1$. The expansion (10.2) becomes

$$G_k(q+it) = 2\zeta(2k) + 2\frac{(-1)^k(2\pi)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{-2\pi nt}(\cos(2\pi kq) + i\sin(2\pi kq)),$$

so the Fourier coefficients of $G_k(q+it)$ are

$$a_n = 2\frac{(-1)^k(2\pi)^{2k}}{(2k-1)!}\sigma_{2k-1}(n)e^{-2\pi nt} \quad \text{and} \quad b_n = ia_n. \tag{10.5}$$

We were unable to evaluate the corresponding Dirichlet series arising from (10.5). Instead we consider the functions

$$G_k^{(\alpha)}(q) := \int_0^{\infty} t^{\alpha}\left[G_k(q+it) - 2\zeta(2k)\right] dt, \tag{10.6}$$

where $\alpha \in \mathbb{R}^+$. We then have the following result.

Example 10.1. The Hurwitz transform of $G_k^{(\alpha)}(q)$ for $\alpha > z + 2k - 2$ is

$$\begin{aligned}\int_0^1 G_k^{(\alpha)}(q)\zeta(z,q)\,dq = 2\pi i\,&\frac{e^{-i\pi z/2}}{\sin(\pi(\alpha - z)/2)}\frac{\Gamma(\alpha+1)\Gamma(1-z)}{\Gamma(2k)\Gamma(3+\alpha-z-2k)}\\ &\times\zeta(2+\alpha-z)\zeta(3+\alpha-z-2k),\end{aligned} \tag{10.7}$$

where $z \in \mathbb{R}_0^-$.

Proof: The Fourier coefficients of $G_k^{(\alpha)}(q)$ are

$$a_n = 2\Gamma(\alpha+1)\frac{(-1)^k(2\pi)^{2k-\alpha-1}}{(2k-1)!}\frac{\sigma_{2k-1}(n)}{n^{\alpha+1}} \quad \text{and} \quad b_n = ia_n. \tag{10.8}$$

The main theorem then yields

$$\int_0^1 G_k^{(\alpha)}(q)\zeta(z,q)\,dq = \frac{2\Gamma(\alpha+1)\Gamma(1-z)(-1)^k ie^{-i\pi z/2}}{(2k-1)!(2\pi)^{2-z-2k+\alpha}}\sum_{n=1}^{\infty}\frac{\sigma_{2k-1}(n)}{n^{2-z+\alpha}}.$$

The last series is identified in [5], page 231, as

$$\sum_{n=1}^{\infty}\frac{\sigma_p(n)}{n^s} = \zeta(s)\zeta(s-p), \quad \operatorname{Re} s > \max\{1, 1 + \operatorname{Re} p\} \tag{10.9}$$

which, for $2 - z + \alpha > 2k$, implies (10.7) after using Riemann's relation (1.34). □

11. A trigonometric example

In this section we compute the Hurwitz transform of powers of sine and cosine, and some related integrals.

Example 11.1. Let $z \in \mathbb{R}_0^-$ and $n \in \mathbb{N}$. Then

$$\int_0^1 \sin^{2n}(\pi q)\,\zeta(z,q)\,dq = \frac{\Gamma(1-z)}{(2\pi)^{1-z}\,2^{2n-1}}\,\sin\left(\frac{\pi z}{2}\right)\sum_{k=1}^{n}\frac{(-1)^k}{k^{1-z}}\binom{2n}{n-k}. \tag{11.1}$$

Example 11.2. Let $m,\ n \in \mathbb{N}_0$. Then

$$\int_0^1 B_{2m+1}(q)\sin^{2n}(\pi q)\,dq = 0, \tag{11.2}$$

and

$$\int_0^1 B_{2m}(q)\sin^{2n}(\pi q)\,dq = \frac{(-1)^{m+1}(2m)!}{2^{2n-1}(2\pi)^{2m}}\sum_{k=1}^{n}\frac{(-1)^k}{k^{2m}}\binom{2n}{n-k}. \tag{11.3}$$

The proof is a direct consequence of results (2.2) and (2.3) for the Fourier coefficients of the Hurwitz zeta function, once we expand $\sin^{2n}(\pi q)$ using a formula of Kogan [17]

$$\sin^{2n} x = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos\left[2(n-k)x\right]. \tag{11.4}$$

Note. Formulae similar to (11.4) exist for other powers of sine and cosine [17]. The same technique used above can be thus applied to obtain the Hurwitz transform of $\sin^{2n+1}(2\pi q)$, $\cos^{2n}(\pi q)$ and $\cos^{2n+1}(2\pi q)$.

Example 11.3. For $n \in \mathbb{N}$

$$\int_0^1 \sin^{2n}(\pi q)\ln\Gamma(q)\,dq = \frac{1}{2^{2n+1}}\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{2n}{n-k} + \frac{1}{2^{2n}}\binom{2n}{n}\ln\sqrt{2\pi}. \tag{11.5}$$

Proof: Simply use (6.7), (11.1) and Wallis' formula

$$\int_0^1 \sin^{2n}(\pi q)\,dq = \frac{1}{2^{2n}}\binom{2n}{n}. \tag{11.6}$$

□

12. The case z positive and polygamma functions

In this section we extend some of the previous formulae to the case $z \in \mathbb{R}^+$ and use them to evaluate the moments of the polygamma functions.

Although the formulae of the previous sections were derived under the assumption $z \leq 0$, so that the Fourier expansion (1.24) could be used, they can be analytically extended to those positive values of z where the integral in question converges. This is so because the Hurwitz transform (1.27) defines an analytic function of z as long as the defining integral converges. For $z > 0$ the only singularity of $\zeta(z, q)$ in the range $0 \leq q \leq 1$ lies actually at $q = 0$, where it behaves as q^{-z}. In fact,

$$\zeta(z, q) = \frac{1}{q^z} + \zeta(z, q+1), \tag{12.1}$$

with $\zeta(z, q)$ finite for $q \geq 1$. The relation (12.1) follows directly from the definition (1.1) of the Hurwitz zeta function when $\text{Re}\, z > 1$ and can be extended to the whole punctured complex z-plane, $\mathbb{C} - \{1\}$, for $q > 0$.

Example 12.1. The formula (3.10) derived in Example 3.5, namely

$$\int_0^1 B_m(q)\zeta(z, q)\, dq = (-1)^{m+1} \frac{m!\, \zeta(z-m)}{(1-z)_m},$$

holds for real $z < 1$ if m equals one or an even integer, and for $z < 2$ otherwise.

Proof: From (1.21) it is seen that near $q = 0$ the Bernoulli polynomials behave as

$$B_m(q) = B_m + mB_{m-1}q + O(q^2), \quad m \geq 1.$$

Thus, the integrand $B_m(q)\zeta(z, q)$ behaves as q^{-z} or q^{1-z}, according if $B_m \neq 0$ or not. The result now follows from the fact that the singularity $q^{-\alpha}$ is integrable for $0 < \alpha < 1$. □

Example 12.2. The formula (3.5) derived in Example 3.2, namely

$$\int_0^1 \zeta^2(z, q)\, dq = 2\Gamma^2(1-z)(2\pi)^{2z-2}\zeta(2-2z)$$

holds for real $z < 1/2$.

Proof: This follows directly from (12.1) and a reasoning similar to the proof of the previous example. □

Example 12.3. Let $n \in \mathbb{N}_0$ and $z \in \mathbb{R}$ such that $n - z + 1 > 0$. Then,

$$\int_0^1 q^n\, \zeta(z, q)\, dq = n! \sum_{k=0}^{n-1} (-1)^k \frac{\zeta(z-k-1)}{(n-k)!(1-z)_{k+1}}. \tag{12.2}$$

Proof: From Theorem 3.7 the equation above holds for $z \le 0$. The integrand $q^n\,\zeta(z,q)$ behaves as q^{n-z} as $q \to 0$, so the integral exists as long as $n - z + 1 > 0$. Now use analytic continuation. □

Based on the result of the last example, we now evaluate the moments of the polygamma functions, defined as

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\psi(z), \quad n \in \mathbb{N}_0, \tag{12.3}$$

where $\psi^{(0)}(z) = \psi(z)$ is the digamma function defined in (1.8).

The polygamma functions can be expressed in terms of the Hurwitz zeta function as (see [27], Chapter 44)

$$\psi^{(m)}(q) = (-1)^{m+1} m!\,\zeta(m+1,q) \quad m = 1, 2, \dots \tag{12.4}$$

and

$$\psi(q) = \lim_{z\to 1}\left[\frac{1}{z-1} - \zeta(z,q)\right]. \tag{12.5}$$

Theorem 12.4. *Let $n, m \in \mathbb{N}$ with $n > m$. Then*

$$\int_0^1 q^n\,\psi^{(m)}(q)\,dq = (-1)^m \frac{n!}{(n-m)!}\left[\frac{\gamma}{n-m+1} + (n-m)!\sum_{k=0}^{m-2}\frac{\Gamma(m-k)\zeta(m-k)}{(n-k)!} + \sum_{k=0}^{n-m-1}(-1)^k\binom{n-m}{k}[H_k\zeta(-k) + \zeta'(-k)]\right]. \tag{12.6}$$

Proof: We compute the limit as $z \to m+1$ in Eq. (12.2). Substitute $z = m + 1 - \varepsilon$ in (12.2) and let $\varepsilon \to 0$. We encounter two types of singularities as $\varepsilon \to 0$: one corresponding to the pole of $\zeta(s)$ at $s = 1$, for $k = m - 1$, and the other corresponding to the vanishing of the Pochhammer symbol $(-m)_{k+1}$, for $k = m, m+1, \dots, n-1$. To derive (12.6) consider the Laurent expansion of (12.2) about $\varepsilon = 0$ up to order ε^0. The following expansions are employed:

$$\begin{aligned}
\zeta(1-\varepsilon) &= \frac{1}{\varepsilon} + \gamma + O(\varepsilon),\\
\zeta(-r-\varepsilon) &= \zeta(-r) - \varepsilon\zeta'(-r) + O(\varepsilon), \quad r = 0, 1, 2, \dots,\\
\Gamma(m+1)\frac{\Gamma(1-\varepsilon)}{\Gamma(m+1-\varepsilon)} &= 1 + H_m\varepsilon + O(\varepsilon),\\
\Gamma(m+1)\frac{\Gamma(-r-\varepsilon)}{\Gamma(m+1-\varepsilon)} &= \frac{(-1)^{r+1}}{r!}\left[\frac{1}{\varepsilon} + H_m - H_r\right] + O(\varepsilon), \quad r = 0, 1, 2, \dots.
\end{aligned} \tag{12.7}$$

A direct calculation yields

$$(-1)^k \frac{\zeta(z-k-1)}{(n-k)!(1-z)_{k+1}}\bigg|_{\substack{z=m+1-\varepsilon \\ k=m-1}} = \frac{1}{m!(n-m-1)!}\left[\frac{1}{\varepsilon} + H_m - \gamma\right],$$

and, for $r = 0, 1, \ldots, n-m-1$,

$$(-1)^k \frac{\zeta(z-k-1)}{(n-k)!(1-z)_{k+1}}\bigg|_{\substack{z=m+1-\varepsilon \\ k=m+r}} = \frac{(-1)^r}{m!r!(n-m+r)!} \times \left[\frac{\zeta(-r)}{\varepsilon} + (H_m - H_r)\,\zeta(-r) - \zeta'(-r)\right].$$

The coefficient of the singular term $1/\varepsilon$ is

$$\frac{1}{m!(n-m)!}\left[\frac{1}{n-m+1} + \sum_{r=0}^{n-m-1} (-1)^r \binom{n-m}{r}\zeta(-r)\right],$$

which vanishes in view of the identity

$$\sum_{r=0}^{j} (-1)^r \binom{j+1}{r}\zeta(-r) = -\frac{1}{j+2}. \tag{12.8}$$

The rest of the terms can be collected to yield (12.6), after multiplying by the overall factor $(-1)^{m+1}m!$ in (12.4). □

Along similar lines, we can use relation (12.5) to prove the following result.

Theorem 12.5. *For $n \in \mathbb{N}$,*

$$\int_0^1 q^n\,\psi(q)\,dq = \zeta'(0) + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}[H_k\zeta(-k) + \zeta'(-k)]. \tag{12.9}$$

Proof: Example 12.3 and (12.5) yield

$$\int_0^1 q^n \psi(q)\,dq = \lim_{z\to 1}\int_0^1 q^n\left[\frac{1}{z-1} - \zeta(z,q)\right]dq$$
$$= \lim_{z\to 1}\frac{1}{z-1}\left[\frac{1}{n+1} + \zeta(z-1) + n!\sum_{k=1}^{n-1}\frac{(-1)^k\zeta(z-k-1)}{(n-k)!(2-z)_k}\right]$$

and (12.9) follows by l'Hopital's rule. □

13. Conclusions

We have evaluated a series of definite integrals whose integrands involve the Hurwitz zeta function.

Most of the formulae involving elementary functions and $\ln\Gamma(q)$ can be considered to be special cases of Theorem 3.1, in view of the relations (1.19), (6.7) and (1.36), the latter written in the form

$$\ln\Gamma(q) + \ln\Gamma(1-q) = \ln\pi - \ln\sin\pi q, \tag{13.1}$$

which respectively relate the Bernoulli polynomials, the logarithm of the gamma function and thus also the function $\ln\sin\pi q$ to $\zeta(z, q)$ in a linear way.

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Notes

1. $\lfloor x \rfloor$ is the floor of x.
2. The factor $m!$ in (4.3) is missing in [25].
3. The sum is over $\mathbb{Z}^2 - (0, 0)$.

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