SOLUTION TO PROBLEM #11519 PROPOSED BY OVIDIU FURDUI

Problem: Find

$$S := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \frac{H_{n+m}}{n+m}$$

where H_n denotes the n-th harmonic number.

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Solution 1: The harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is given by

$$H_n = \int_0^1 \frac{x^n - 1}{x - 1}\, dx.$$

Therefore the requested sum is

$$S = \sum_{n,m} \frac{(-1)^{n+m}}{n+m} \int_0^1 \frac{x^{n+m} - 1}{x - 1}\, dx.$$

To evaluate it in closed form, define

$$S(b) = \sum_{n,m} \frac{(-1)^{n+m}}{n+m} \int_0^1 \frac{(x^{n+m} - 1) b^{n+m}}{x - 1}\, dx$$

and observe that $S = S(1)$. Differentiating with respect to the parameter b gives

$$\begin{aligned} S'(b) &= \frac{1}{b} \sum_{n,m \geq 1} (-1)^{n+m} \int_0^1 \frac{(x^{n+m} - 1) b^{n+m}}{x - 1}\, dx \\ &= \frac{1}{b} \int_0^1 \left[\sum_{n,m \geq 1} (-xb)^{n+m} - (-b)^{n+m} \right] \frac{dx}{x - 1}. \end{aligned}$$

The double series are easy to compute: they are simply squares of geometric series. Thus,

$$S'(b) = \frac{b}{(1+b)^2} \int_0^1 \frac{1 + x(1+2b)}{(1+bx)^2}\, dx. \tag{1}$$

Evaluating the elementary integral yields

$$S'(b) = -\frac{1}{(1+b)^2} + \frac{2\ln(1+b)}{(1+b)^2} + \frac{\ln(1+b)}{b(1+b)^2}.$$

Finally, integrating with respect to b (done by using `Mathematica`, but a *computer-free* proof is also possible) yields

$$S(b) = -\frac{\ln(1+b)}{1+b} - \frac{1}{2} \ln^2(1+b) - \mathrm{Dilog}(-b),$$

where

$$\mathrm{Dilog}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^2}$$

is the *dilogarithm* function. The special case $b = 1$, using $\mathrm{Dilog}(-1) = -\pi^2/12$ yields

$$S(1) = \frac{\pi^2}{12} - \frac{\ln 2}{2} - \frac{\ln^2 2}{2}.$$

Solution 2: An alternative approach to this problem begins with the generalization

$$T(q) := \sum_{n,m\geq 1} q^{n+m} \frac{H_{n+m}}{n+m}.$$

A closed-form expression is now derived for T.

Let $k = n + m$ to reduce $T(q)$ to a single sum

$$T(q) = \sum_{k\geq 2} q^k \frac{(k-1)\, H_k}{k} = \sum_{k\geq 1} q^k H_k - \sum_{k\geq 1} q^k \frac{H_k}{k}. \tag{2}$$

For $|q| < 1$, the uniformly convergent series

$$-\ln(1-q) = \sum_{k\geq 1} \frac{q^k}{k} \text{ and } \frac{1}{1-q} = \sum_{k\geq 1} q^{k-1}$$

give

$$-\frac{\ln(1-q)}{1-q} = \sum_{k\geq 1} q^k H_k$$

by Cauchy's product formula. Thus

$$\begin{aligned} T(q) &= -\frac{\ln(1-q)}{1-q} + \int \frac{\ln(1-q)}{1-q}\, dq + \int \frac{\ln(1-q)}{q}\, dq \\ &= -\frac{\ln(1-q)}{1-q} - \frac{1}{2}\ln^2(1-q) - \sum_{k\geq 1} \frac{q^k}{k^2}. \end{aligned}$$

The limit $q \to -1^+$ implies the result:

$$T(-1) = -\frac{1}{2}\ln 2 - \frac{1}{2}\ln^2 2 + \frac{1}{12}\pi^2$$

since

$$\sum_{k\geq 1} \frac{(-1)^{k-1}}{k^2} = \frac{1}{2}\zeta(2) = \frac{\pi^2}{12}.$$