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The integrals in Gradshteyn and Ryzhik. Part 23: Combination of logarithms and rational functions

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ABSTRACT. The table of Gradshteyn and Ryzhik contains many entries where the integrand is a combination of a rational function and a logarithmic function. The proofs presented here, complete the evaluation of all entries in Section 4.231 and 4.291.

1. Introduction

The table of integrals [**6**] contains many entries of the form

$$\int_a^b R_1(x) \ln R_2(x) \, dx \tag{1.1}$$

where R_1 and R_2 are rational functions. Some of these examples have appeared in previous papers: entry **4.291.1**

$$\int_0^1 \frac{\ln(1+x)}{x} \, dx = \frac{\pi^2}{12} \tag{1.2}$$

as well as entry **4.291.2**

$$\int_0^1 \frac{\ln(1-x)}{x} \, dx = -\frac{\pi^2}{6} \tag{1.3}$$

have been established in [**4**], entry **4.212.7**

$$\int_1^e \frac{\ln x \, dx}{(1+\ln x)^2} = \frac{e}{2} - 1 \tag{1.4}$$

appears in [**2**] and entry **4.231.11**

$$\int_0^a \frac{\ln x \, dx}{x^2+a^2} = \frac{\pi \ln a}{4a} - \frac{G}{a}, \tag{1.5}$$

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where

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{1.6}$$

is the Catalan constant, has appeared in [**5**]. The value of entry **4.233.1**

$$\int_0^1 \frac{\ln x\, dx}{x^2+x+1} = \frac{2}{9}\left[\frac{2\pi^2}{3} - \psi'\left(\frac{1}{3}\right)\right], \tag{1.7}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, was established in [**8**].

A standard trick employed in the evaluations of integrals over $[0,\infty)$, is to transform the interval $[1,\infty)$ back to $[0,1]$ via $t = 1/x$. This gives

$$\int_0^\infty R(x)\, \ln x\, dx = \int_0^1 \left[R(x) - \frac{1}{x^2} R\left(\frac{1}{x}\right)\right]\, dx. \tag{1.8}$$

In particular, if the rational function satisfies

$$R\left(\frac{1}{x}\right) = x^2 R(x), \tag{1.9}$$

then

$$\int_0^\infty R(x)\, \ln x\, dx = 0. \tag{1.10}$$

This is the case for $R(x) = \dfrac{1+x^2}{(1-x^2)^2}$ and (1.10) appears as entry **4.234.3** in [**6**].

The goal of this paper is to present a sytematic evaluation of the entries in [**6**] of the form (1.1).

2. Combinations of logarithms and linear rational functions

Example 2.1. Entry **4.291.3** states that

$$\int_0^{1/2} \frac{\ln(1-x)}{x}\, dx = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}. \tag{2.1}$$

To evaluate this integral let $t = -\ln(1-x)$ to produce

$$\int_0^{1/2} \frac{\ln(1-x)}{x}\, dx = -\int_0^{\ln 2} \frac{te^{-t}\, dt}{1-e^{-t}}. \tag{2.2}$$

This last integral can be written as

$$\int_0^{\ln 2} t\, dt - \int_0^{\ln 2} \frac{t\, dt}{1-e^{-t}}. \tag{2.3}$$

The first integral is elementary and has value $\frac{1}{2}\ln^2 2$. The second integral was evaluated as $\pi^2/12$ in [**3**].

EXAMPLE 2.2. The change of variables $t = x/2$ converts (2.1) to

$$\int_0^{1/2} \ln\left(1 - \frac{t}{2}\right) \frac{dt}{t} = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}. \tag{2.4}$$

This is entry **4.291.4** of [**6**].

EXAMPLE 2.3. Entry **4.291.5** states that

$$\int_0^1 \ln\left(\frac{1+x}{2}\right) \frac{dx}{1-x} = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}. \tag{2.5}$$

To evaluate this entry, let $u = (1-x)/2$ to reduce it to (2.1)

EXAMPLE 2.4. Differentiating

$$\int_0^1 (1+x)^{-a}\, dx = \frac{2^{-a}(2^a - 2)}{a-1} \tag{2.6}$$

with respect to a gives

$$\int_0^1 (1+x)^{-a} \ln(1+x)\, dx = \frac{1}{(a-1)^2}\left(2^{-a}(-2 + 2^a + 2\ln 2 - 2a\ln 2\right). \tag{2.7}$$

Now let $a \to 1$ to obtain

$$\int_0^1 \frac{\ln(1+x)}{1+x}\, dx = \frac{1}{2}\ln^2 2. \tag{2.8}$$

This is entry **4.291.6**.

EXAMPLE 2.5. The partial fraction decomposition

$$\frac{1}{x(1+x)} = \frac{1}{x} - \frac{1}{1+x} \tag{2.9}$$

gives

$$\int_0^1 \frac{\ln(1+x)}{x(1+x)}\, dx = \int_0^1 \frac{\ln(1+x)}{x}\, dx - \int_0^1 \frac{\ln(1+x)}{1+x}. \tag{2.10}$$

The first integral is entry **4.291.1** and it has value $\pi^2/12$ as shown in [**4**]. The second integral is $\frac{1}{2}\ln^2 2$ as established in Example 2.4. This gives entry **4.291.12**

$$\int_0^1 \frac{\ln(1+x)}{x(1+x)}\, dx = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2. \tag{2.11}$$

EXAMPLE 2.6. Entry **4.291.13** is

$$\int_0^\infty \frac{\ln(1+x)\, dx}{x(1+x)} = \frac{\pi^2}{6}. \tag{2.12}$$

Split the integral over $[0,1]$ and $[1,\infty)$ and make the change of variables $t = 1/x$ in the second part. This gives

$$\int_0^\infty \frac{\ln(1+x)\, dx}{x(1+x)} = \int_0^1 \frac{\ln(1+x)\, dx}{x(1+x)} + \int_0^1 \frac{\ln(1+t) - \ln t}{1+t}\, dt. \tag{2.13}$$

Expand the first integral in partial fractions to obtain

$$\int_0^\infty \frac{\ln(1+x)\,dx}{x(1+x)} = \int_0^1 \frac{\ln(1+x)}{x}\,dx - \int_0^1 \frac{\ln x}{1+x}\,dx. \tag{2.14}$$

Integrate by parts the second integral to obtain

$$\int_0^\infty \frac{\ln(1+x)\,dx}{x(1+x)} = 2\int_0^1 \frac{\ln(1+x)}{x}\,dx. \tag{2.15}$$

The evaluation

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12} \tag{2.16}$$

that appears as entry **4.291.1** has been established in [**4**].

3. Combinations of logarithms and rational functions with denominators that are squares of linear terms

This section evaluates integrals of the form

$$\int_a^b R_2(x)\,\ln R_1(x)\,dx \tag{3.1}$$

where R_1, R_2 are rational functions and the denominator of R_2 is a quadratic polynomial of the form $(cx+d)^2$.

Example 3.1. Entry **4.291.14** is

$$\int_0^1 \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{1}{a(a-b)}\ln\frac{a+b}{b} + \frac{2\ln 2}{b^2-a^2} \tag{3.2}$$

and

$$\int_0^1 \frac{\ln(1+x)\,dx}{(x+1)^2} = \frac{1-\ln 2}{2} \tag{3.3}$$

gives the value when $a=b$, after scaling.

To evaluate the first case, integrate by parts to get

$$\int_0^1 \frac{\ln(1+x)}{(ax+b)^2}\,dx = -\frac{\ln 2}{a(a+b)} + \frac{1}{a}\int_0^1 \frac{dx}{(1+x)(ax+b)}. \tag{3.4}$$

The result now follows by expanding the second integrand in partial fractions.

The case $a=b$ is obtained by a direct integration by parts:

$$\int_0^1 \frac{\ln(1+x)}{(1+x)^2}\,dx = -\frac{\ln 2}{2} + \int_0^1 \frac{dx}{(1+x)^2}. \tag{3.5}$$

This last integral is $1/2$ and the result has been established.

The same procedure gives entry **4.291.20**:

$$\int_0^1 \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{2(a-b)}\left[(a+b)\ln(a+b) - 2b\ln b - 2a\ln 2\right], \tag{3.6}$$

for $a \neq b$.

EXAMPLE 3.2. The partial fraction decomposition

$$\frac{1-x^2}{(ax+b)^2\,(bx+a)^2} = \frac{1}{a^2-b^2}\left[\frac{1}{(ax+b)^2} - \frac{1}{(bx+a)^2}\right] \tag{3.7}$$

and Example 3.1 gives the evaluation of entry **4.291.25**:

$$\begin{aligned}\int_0^1 \frac{(1-x^2)\ln(1+x)\,dx}{(ax+b)^2\,(bx+a)^2} &= \frac{1}{(a^2-b^2)(a-b)}\left[\frac{a+b}{ab}\ln(a+b) - \frac{\ln b}{a} - \frac{\ln a}{b}\right] \\ &\quad - \frac{4\ln 2}{(a^2-b^2)^2}.\end{aligned}$$

The answer may be written in the more compact form

$$\frac{-a^2\ln a - b\,[b\ln b + a\ln(16ab)] + (a+b)^2\ln(a+b)}{ab(a^2-b^2)^2}, \tag{3.8}$$

but this form hiddes the symmetry of the integral.

EXAMPLE 3.3. Entry **4.291.15** is

$$\int_0^\infty \frac{\ln(1+x)\,dx}{(ax+b)^2} = \frac{\ln a - \ln b}{a(a-b)} \tag{3.9}$$

for $a \neq b$. In the case $a = b$, the integral scales to

$$\int_0^\infty \frac{\ln(1+x)\,dx}{(1+x)^2} = 1. \tag{3.10}$$

To evaluate this entry, integrate by parts to obtain

$$\int_0^\infty \frac{\ln(1+x)\,dx}{(ax+b)^2} = \frac{1}{a}\int_0^\infty \frac{dx}{(1+x)(ax+b)}. \tag{3.11}$$

This last integral is evaluated by using the partial fraction decomposition

$$\frac{1}{(1+x)(ax+b)} = \frac{1}{b-a}\left(\frac{1}{1+x} - \frac{a}{ax+b}\right). \tag{3.12}$$

Integration by parts in the case $a = b$ (taken to be 1 by scaling) gives

$$\int_0^\infty \frac{\ln(1+x)\,dx}{(1+x)^2} = \int_0^\infty \frac{dx}{(1+x)^2} = 1. \tag{3.13}$$

The same procedure gives entry **4.291.21**:

$$\int_0^\infty \frac{\ln(ax+b)\,dx}{(1+x)^2} = \frac{a\ln a - b\ln b}{a-b}. \tag{3.14}$$

for $a \neq b$. The value of entry **4.291.17**:

$$\int_0^\infty \frac{\ln(a+x)}{(b+x)^2}\,dx = \frac{a\ln a - b\ln b}{b(a-b)} \tag{3.15}$$

is obtained from (3.14) by the change of variables $x = bt$.

EXAMPLE 3.4. The partial fraction decomposition (3.7) given in Example 3.2 produces the value of entry **4.291.26**

$$\int_0^\infty \frac{(1-x^2)\ln(1+x)\,dx}{(ax+b)^2\,(bx+a)^2} = \frac{\ln b - \ln a}{ab(a^2-b^2} \tag{3.16}$$

form Example 3.3.

4. Combinations of logarithms and rational functions with quadratic denominators

This section considers integrals of the form (1.1) where the denominator of $R_2(x)$ is a polynomial of degree 2 with non-real roots.

EXAMPLE 4.1. Entry **4.291.8** states that

$$\int_0^1 \frac{\ln(1+x)\,dx}{1+x^2} = \frac{\pi}{8}\ln 2. \tag{4.1}$$

The proof of this evaluation is based on some entries of [**6**] that have been established in [**4**]. The reader is invited to provide a direct proof.

The change of variables $x = \tan\varphi$ gives

$$\begin{aligned} \int_0^1 \frac{\ln(1+x)\,dx}{1+x^2} &= \int_0^{\pi/4} \ln(1+\tan\varphi)d\varphi \\ &= \int_0^{\pi/4} \ln(\sin\varphi+\cos\varphi)d\varphi - \int_0^{\pi/4} \ln\cos\varphi\,d\varphi. \end{aligned}$$

The value

$$\int_0^{\pi/4} \ln(\sin\varphi+\cos\varphi)d\varphi = -\frac{\pi}{8}\ln 2 + \frac{G}{2}$$

is entry **4.225.2** and

$$\int_0^{\pi/4} \ln\cos\varphi\,d\varphi = -\frac{\pi}{4}\ln 2 + \frac{G}{2}$$

is entry **4.224.5**. Both examples are evaluated in [**4**]. This gives the result.

The same technique gives entry **4.291.10**

$$\int_0^1 \frac{\ln(1-x)\,dx}{1+x^2} = \frac{\pi}{8}\ln 2 - G. \tag{4.2}$$

This time, entry **4.225.1**

$$\int_0^{\pi/4} \ln(\cos\varphi-\sin\varphi)d\varphi = -\frac{\pi}{8}\ln 2 - \frac{G}{2}$$

is employed.

EXAMPLE 4.2. Entry **4.291.9**

$$\int_0^\infty \frac{\ln(1+x)\,dx}{1+x^2} = \frac{\pi}{4}\ln 2 + G \tag{4.3}$$

is equivalent, via $x = \tan\varphi$, to the identity

$$\int_0^{\pi/2} \ln(\sin\varphi + \cos\varphi) d\varphi - \int_0^{\pi/2} \ln\cos\varphi \, d\varphi = \frac{\pi}{4}\ln 2 + G. \tag{4.4}$$

The first integral is entry **4.225.2** and it has the value $-\frac{1}{4}\pi\ln 2 + G$; the second integral is entry **4.224.6** with value $-\frac{1}{2}\pi\ln 2$. Both of these examples have been established in [**4**].

Example 4.3. The change of variables $t = 1/x$ gives

$$\int_1^\infty \frac{\ln(x-1)\,dx}{1+x^2} = \int_0^1 \frac{\ln(1-t)\,dt}{1+t^2} - \int_0^1 \frac{\ln t\,dt}{1+t^2}. \tag{4.5}$$

The first integral has the value $\frac{1}{8}\pi\ln 2 - G$ and it appears as entry **4.291.10** (it has been established as (4.2)). The second integral is the special case $a = 1$ of (1.5). This gives the value of entry **4.291.11**:

$$\int_1^\infty \frac{\ln(x-1)\,dx}{1+x^2} = \frac{\pi}{8}\ln 2. \tag{4.6}$$

Example 4.4. A small number of entries in [**6**] can be evaluated from entry **4.231.9**

$$\int_0^\infty \frac{\ln x\,dx}{x^2+q^2} = \frac{\pi}{2}\frac{\ln q}{q}, \tag{4.7}$$

evaluated in [**4**]. Expanding in partial fractions gives the identity

$$\int_0^\infty \frac{\ln x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(b^2-a^2)}\left(\frac{\ln a}{a} - \frac{\ln b}{b}\right). \tag{4.8}$$

This provides the evaluation of entry **4.234.6**

$$\int_0^\infty \frac{\ln x\,dx}{(a^2+b^2x^2)(1+x^2)} = \frac{\pi b}{2a(b^2-a^2)}\ln\frac{a}{b} \tag{4.9}$$

via the relation

$$\int_0^\infty \frac{\ln x\,dx}{(a^2+b^2x^2)(1+x^2)} = \frac{1}{b^2}\int_0^\infty \frac{\ln x\,dx}{(x^2+a^2/b^2)(x^2+1)}, \tag{4.10}$$

entry **4.234.7**

$$\int_0^\infty \frac{\ln x\,dx}{(x^2+a^2)(1+b^2x^2)} = \frac{\pi}{2(1-a^2b^2)}\left(\frac{\ln a}{a} + b\ln b\right) \tag{4.11}$$

via the relation

$$\int_0^\infty \frac{\ln x\,dx}{(x^2+a^2)(1+b^2x^2)} = \frac{1}{b^2}\int_0^\infty \frac{\ln x\,dx}{(x^2+a^2)(x^2+1/b^2)}, \tag{4.12}$$

and finally, entry **4.234.8**

$$\int_0^\infty \frac{x^2\,\ln x\,dx}{(a^2+b^2x^2)(1+x^2)} = \frac{\pi a}{2b(b^2-a^2)}\ln\frac{b}{a} \tag{4.13}$$

using the partial fraction decomposition

$$\frac{x^2}{(a^2+b^2x^2)(1+x^2)} = \frac{1}{(b^2-a^2)}\frac{1}{x^2+1} - \frac{a^2}{b^2(b^2-a^2)}\frac{1}{x^2+a^2/b^2}. \tag{4.14}$$

The details are left to the reader.

5. An example via recurrences

The integral

$$F_n(s) = \int_0^1 x^n(1+x)^s\,dx \tag{5.1}$$

for $n \in \mathbb{N}$ and $s \in \mathbb{R}$, is integrated by parts (with $u = x^n(x+1)$ and $dv = (x+1)^{s-1}\,dx$), to produce the recurrence

$$F_n(s) = \frac{2^{s+1}}{n+s+1} - \frac{n}{n+s+1}F_{n-1}(s). \tag{5.2}$$

The initial condition is

$$F_0(s) = \int_0^1 (x+1)^s\,dx = \frac{2^{s+1}-1}{s+1}. \tag{5.3}$$

The recurrence permits the evaluation of $F_n(s)$, for any fixed $n \in \mathbb{N}$. For instance,

$$\begin{aligned} F_1(s) &= \frac{s2^{s+1}+1}{(s+1)(s+2)} \\ F_2(s) &= \frac{2\left[2^s(s^2+s+2)-1\right]}{(s+1)(s+2)(s+3)} \\ F_3(s) &= \frac{2\left[2^s(s^3+3s^2+8s)+3\right]}{(s+1)(s+2)(s+3)(s+4)}. \end{aligned}$$

Differentiating (5.2) produces a recurrence for

$$G_n(s) = \int_0^1 \frac{x^n\,\ln(1+x)}{(1+x)^s}\,dx \tag{5.4}$$

in the form

$$\begin{aligned} G_n(s) &= -\frac{2^{1-s}}{(n+1-s)^2} + \frac{2^{1-s}\,\ln 2}{n+1-s} \\ &\quad + \frac{n}{(n-s+1)^2}F_n(-s) - \frac{n}{n-s+1}G_{n-1}(s). \end{aligned} \tag{5.5}$$

This produces the value of $G_n(s)$, starting from

$$G_0(s) = \int_0^1 \frac{\ln(1+x)}{(1+x)^s}\,dx = \frac{2^{1-s}\,\ln 2}{1-s} - \frac{2^{1-s}-1}{(1-s)^2}. \tag{5.6}$$

For example,

$$G_1(s) = \frac{2^s(2s-3) - 2\ln 2s^3 + 2(3\ln 2 - 1)s^2 - 4\ln 2s + 4}{2^s(s-1)^2(s-2)^2}. \tag{5.7}$$

EXAMPLE 5.1. Entry **4.291.23** in [**6**] states that

$$\int_0^1 \ln(1+x)\,\frac{1+x^2}{(1+x)^4}\,dx = -\frac{\ln 2}{3} + \frac{23}{72}. \tag{5.8}$$

This corresponds to the value $G_0(4) + G_2(4)$. The recurrence (5.5) gives the required data to verify this entry.

6. An elementary example

Integrals of the form

$$\int_a^b \ln R_1(x) \frac{d}{dx} R_2(x)\,dx \tag{6.1}$$

for rational functions R_1, R_2 can be reduced to the integration of a rational function. Indeed, integration by parts yields

$$\int_a^b \ln R_1(x) \frac{d}{dx} R_2(x)\,dx = \text{ boundary terms} - \int_a^b R_3(x)\,dx \tag{6.2}$$

with $R_3 = R_1' R_2 / R_1$.

EXAMPLE 6.1. Entry **4.291.27** states that

$$\int_0^1 \ln(1+ax)\,\frac{1-x^2}{(1+x^2)^2}\,dx = \frac{(1+a)^2}{1+a^2}\frac{\ln(1+a)}{2} - \frac{\ln 2}{2}\frac{a}{1+a^2} - \frac{\pi}{4}\frac{a^2}{1+a^2}. \tag{6.3}$$

This example fits the pattern described above, since

$$\frac{1-x^2}{(1+x^2)^2} = \frac{d}{dx}\frac{x}{1+x^2}. \tag{6.4}$$

Therefore

$$\begin{aligned} \int_0^1 \ln(1+ax)\,\frac{1-x^2}{(1+x^2)^2}\,dx &= \int_0^1 \ln(1+ax)\,\frac{d}{dx}\frac{x}{1+x^2}\,dx \\ &= \frac{\ln(1+a)}{2} - a\int_0^1 \frac{x\,dx}{(1+x^2)(1+ax)}. \end{aligned}$$

The partial fraction decomposition

$$\frac{x}{(1+x^2)(1+ax)} = -\frac{a}{1+a^2}\frac{1}{1+ax} + \frac{a}{1+a^2}\frac{1}{1+x^2} + \frac{1}{1+a^2}\frac{x}{1+x^2}$$

and the evaluation of the remaining elementary integrals completes the solution to this problem.

EXAMPLE 6.2. Entry **4.291.28**

$$\int_0^\infty \ln(a+x)\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \frac{1}{a^2+b^2}\left(a\ln\frac{b}{a} - \frac{\pi b}{2}\right) \tag{6.5}$$

also fits the pattern in this section since

$$\frac{d}{dx}\frac{x}{x^2+b^2} = \frac{b^2-x^2}{(b^2+x^2)^2}. \tag{6.6}$$

Integrating by parts and checking that the boundary terms vanish, produces

$$\int_0^\infty \ln(a+x)\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = -\int_0^\infty \frac{x\,dx}{(x^2+b^2)(x+a)}. \tag{6.7}$$

It is convenient to introduce the scaling $x = bt$ to transform the last integral to

$$\int_0^\infty \frac{x\,dx}{(x^2+b^2)(x+a)} = \frac{1}{b}\int_0^\infty \frac{t\,dt}{(1+t^2)(t+c)} \tag{6.8}$$

with $c = a/b$. The evaluation is completed using the partial fraction decomposition

$$\frac{t}{(t^2+1)(t+c)} = -\frac{c}{c^2+1}\frac{1}{t+c} + \frac{1}{1+c^2}\frac{1}{t^2+1} + \frac{c}{c^2+1}\frac{t}{t^2+1}$$

and integrating from $t=0$ to $t=N$ and taking the limit as $N \to \infty$. The reader will easily check that the divergent pieces, coming from $1/(t+c)$ and $t/(t^2+1)$ cancel out.

Example 6.3. Entry **4.291.29** appears as

$$\int_0^\infty \ln^2(a-x)\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \frac{2}{a^2+b^2}\left(a\ln\frac{a}{b} - \frac{\pi b}{2}\right) \tag{6.9}$$

but it should be written as

$$\int_0^\infty \ln\left[(a-x)^2\right]\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \frac{2}{a^2+b^2}\left(a\ln\frac{a}{b} - \frac{\pi b}{2}\right). \tag{6.10}$$

This is a singular integral and the value should be interpreted as a Cauchy principal value

$$\int_0^\infty \ln\left[(a-x)^2\right]\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \lim_{\varepsilon\to 0}\int_0^{a-\varepsilon} \ln\left[(a-x)^2\right]\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx + \int_{a+\varepsilon}^\infty \ln\left[(a-x)^2\right]\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx.$$

The first integral is

$$\begin{aligned}\int_0^{a-\varepsilon} \ln\left[(a-x)^2\right]\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx &= \int_0^{a-\varepsilon} 2\ln(a-x)\,\frac{d}{dx}\frac{x}{x^2+b^2}\,dx \\ &= \frac{2(a-\varepsilon)}{(a-\varepsilon)^2+b^2}\ln\varepsilon + \int_0^{a-\varepsilon}\frac{2x\,dx}{(a-x)(x^2+b^2)},\end{aligned}$$

after integration by parts. The second integral produces

$$\begin{aligned}\int_{a+\varepsilon}^\infty \ln\left[(a-x)^2\right]\,\frac{b^2-x^2}{(b^2+x^2)^2}\,dx &= \int_{a+\varepsilon}^\infty 2\ln(x-a)\,\frac{d}{dx}\frac{x}{x^2+b^2}\,dx \\ &= -\frac{2(a+\varepsilon)}{(a+\varepsilon)^2+b^2}\ln\varepsilon + \int_{a+\varepsilon}^\infty\frac{2x\,dx}{(x-a)(x^2+b^2)}.\end{aligned}$$

The reader will ckeck that the boundary terms vanish as $\varepsilon \to 0$. This produces

$$\int_0^\infty \ln\left[(a-x)^2\right] \frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \lim_{\varepsilon\to 0}\int_0^{a-\varepsilon} \frac{2x\,dx}{(a-x)(x^2+b^2)} + \int_{a+\varepsilon}^\infty \frac{2x\,dx}{(a-x)(x^2+b^2)}. \tag{6.11}$$

The partial fraction decomposition

$$\frac{2x}{(a-x)(x^2+b^2)} = -\frac{2a}{a^2+b^2}\frac{1}{x-a} - \frac{2b}{a^2+b^2}\frac{b}{x^2+b^2} + \frac{a}{a^2+b^2}\frac{2x}{x^2+b^2} \tag{6.12}$$

gives

$$\int_0^{a-\varepsilon} \frac{2x\,dx}{(a-x)(x^2+b^2)} = \frac{2a}{a^2+b^2}\left[\ln a - \ln\varepsilon\right] - \frac{2b}{a^2+b^2}\tan^{-1}\frac{a-\varepsilon}{b} + \frac{a}{a^2+b^2}\left[\ln[(a-\varepsilon)^2+b^2] - 2\ln b\right].$$

A similar computation yields

$$\int_{a+\varepsilon}^N \frac{2x\,dx}{(a-x)(x^2+b^2)} = \frac{a}{a^2+b^2}\left\{\ln(N^2+b^2) - 2\ln(N-a) + 2\ln\varepsilon - \ln\left[(a+\varepsilon)^2+b^2\right]\right\} + \frac{2b}{a^2+b^2}\left[\tan^{-1}\left(\frac{a+\varepsilon}{b}\right) - \tan^{-1}\left(\frac{N}{b}\right)\right].$$

Now let $N \to \infty$ and use $\ln(N^2+b^2) - 2\ln(N-a) \to 0$ to obtain

$$\int_{a+\varepsilon}^\infty \frac{2x\,dx}{(a-x)(x^2+b^2)} = \frac{a}{a^2+b^2}\left\{2\ln\varepsilon - \ln\left[(a+\varepsilon)^2+b^2\right]\right\} + \frac{2b}{a^2+b^2}\left[\tan^{-1}\left(\frac{a+\varepsilon}{b}\right) - \frac{\pi}{2}\right].$$

Observe that the singular terms in (6.11), namely those containing the factor $\ln\varepsilon$, cancel out. The remaining terms produce the stated answer as $\varepsilon \to 0$. This completes the evaluation.

Example 6.4. Entry **4.291.30** written as

$$\int_0^\infty \ln\left[(a-x)^2\right] \frac{x\,dx}{(b^2+x^2)^2} = \frac{1}{a^2+b^2}\left(\ln b - \frac{\pi a}{2b} + \frac{a^2}{b^2}\ln a\right) \tag{6.13}$$

is evaluated as Example 6.3. Start with the identity

$$\frac{d}{dx}\left(-\frac{1}{2(x^2+b^2)}\right) = \frac{x}{(x^2+b^2)^2} \tag{6.14}$$

and then proceed as before. The details are elementary and they are left to the reader.

7. Some parametric examples

This section considers some entries of [**6**] that depend on a parameter.

EXAMPLE 7.1. Entry **4.291.18** states that

$$\int_0^a \frac{\ln(1+ax)\,dx}{1+x^2} = \frac{1}{2}\tan^{-1} a \, \ln(1+a^2). \tag{7.1}$$

Differentiating the left-hand side with respect to a gives

$$\frac{\ln(1+a^2)}{1+a^2} + \int_0^a \frac{x\,dx}{(1+ax)(1+x^2)}. \tag{7.2}$$

The verification of this entry will start with the evaluation of the rational integral

$$R(a) := \int_0^a \frac{x\,dx}{(1+ax)(1+x^2)}. \tag{7.3}$$

The partial fraction decomposition

$$\frac{x}{(1+ax)(1+x^2)} = -\frac{1}{1+a^2}\frac{a}{1+ax} + \frac{a}{1+a^2}\frac{1}{1+x^2} + \frac{1}{2(1+a^2)}\frac{2x}{1+x^2} \tag{7.4}$$

gives

$$R(a) = -\frac{\ln(1+a^2)}{1+a^2} + \frac{a}{1+a^2}\tan^{-1} a + \frac{\ln(1+a^2)}{2(1+a^2)}. \tag{7.5}$$

Motivated by the expression in the entry being evaluated, observe that

$$\int_0^a \frac{x\,dx}{(1+ax)(1+x^2)} + \frac{\ln(1+a^2)}{1+a^2} = \frac{1}{2}\frac{d}{da}\left[\tan^{-1} a \, \ln(1+a^2)\right]. \tag{7.6}$$

Now integrate this identity from 0 to a to obtain

$$\int_0^a \left[\int_0^b \frac{x\,dx}{(1+bx)(1+x^2)} + \frac{\ln(1+b^2)}{1+b^2} \right] db + \int_0^a \frac{\ln(1+b^2)}{1+b^2}\,db = \frac{1}{2}\tan^{-1} a \, \ln(1+a^2). \tag{7.7}$$

Exchange the order of integration to produce

$$\begin{aligned} \int_0^a \int_0^b \frac{x\,dx}{(1+bx)(1+x^2)}\,db &= \int_0^a \frac{x}{1+x^2} \int_x^a \frac{db}{1+bx}\,dx \\ &= \int_0^a \frac{1}{1+x^2}\left[\ln(1+ax) - \ln(1+x^2)\right] dx. \end{aligned}$$

The result now follows from (7.7).

EXAMPLE 7.2. Entry **4.291.16** states that

$$\int_0^1 \frac{\ln(a+x)\,dx}{a+x^2} = \frac{1}{2\sqrt{a}}\cot^{-1}\sqrt{a}\,\ln[a(1+a)]. \tag{7.8}$$

The change of variables $x = \sqrt{a}t$ gives

$$\int_0^1 \frac{\ln(a+x)\,dx}{a+x^2} = \frac{1}{\sqrt{a}} \left[\ln a \int_0^{1/\sqrt{a}} \frac{dt}{1+t^2} + \int_0^{1/\sqrt{a}} \frac{\ln(1+t/\sqrt{a})}{1+t^2}\,dt \right]. \tag{7.9}$$

The first integral is elementary and the second one corresponds to (7.1).

EXAMPLE 7.3. Entry **4.291.19** states that

$$\int_0^1 \frac{\ln(1+ax)\,dx}{1+ax^2} = \frac{1}{2\sqrt{a}} \tan^{-1}\sqrt{a}\,\ln(1+a). \tag{7.10}$$

This follows directly from (7.1) by the change of variables $x = t/\sqrt{a}$ and replacing a by $\sqrt{a}$.

EXAMPLE 7.4. Entry **4.291.7** is the identity

$$\int_0^\infty \frac{\ln(1+ax)\,dx}{1+x^2} = \frac{\pi}{4}\ln(1+a^2) - \int_0^a \frac{\ln u\,du}{1+u^2}. \tag{7.11}$$

Differentiating the left-hand side gives

$$\begin{aligned} \frac{d}{da}\int_0^\infty \frac{\ln(1+ax)\,dx}{1+x^2} &= \int_0^\infty \frac{x\,dx}{(1+ax)(1+x^2)} \\ &= \frac{\pi}{2}\frac{a}{1+a^2} - \frac{\ln a}{1+a^2}, \end{aligned}$$

where the last evaluation is established by partial fractions. The result now follows by integrating back with respect to a.

REMARK 7.1. The current version of `Mathematica` gives

$$\int_0^a \frac{\ln x\,dx}{1+x^2} = \tan^{-1} a\,\ln a - \frac{i}{2}\text{PolyLog}[2,-ia] + \frac{i}{2}\text{PolyLog}[2,ia]$$

but is unable to provide an analytic expression for the integral

$$\int_0^\infty \frac{\ln(1+ax)\,dx}{1+x^2}.$$

Entries of [**6**] that can be evaluated in terms of polylogarithms will be described in a future publication.

EXAMPLE 7.5. Entry **4.291.24** states that

$$\int_0^1 \frac{(1+x^2)\,\ln(1+x)}{(a^2+x^2)(1+a^2x^2)}\,dx = \frac{1}{2a(1+a^2)}\left[\frac{\pi}{2}\ln(1+a^2) - 2\tan^{-1} a\,\ln a\right].$$

The evaluation of this entry starts with the partial fraction decomposition

$$\frac{1+x^2}{(a^2+x^2)(1+a^2x^2)} = \frac{1}{1+a^2}\left[\frac{1}{x^2+a^2} + \frac{1}{1+a^2x^2}\right] \tag{7.12}$$

that yields the identity

$$\int_0^1 \frac{(1+x^2)\,\ln(1+x)}{(a^2+x^2)(1+a^2x^2)}\,dx = \frac{1}{1+a^2}\left[\int_0^1 \frac{\ln(1+x)\,dx}{x^2+a^2} + \int_0^1 \frac{\ln(1+x)\,dx}{1+a^2x^2}\right],$$

and the change of variables $t = 1/x$ then produces

$$\int_0^1 \frac{\ln(1+x)\,dx}{1+a^2x^2} = \int_1^\infty \frac{\ln(1+t)\,dt}{t^2+a^2} - \int_1^\infty \frac{\ln t\,dt}{t^2+a^2}.$$

Therefore

$$\int_0^1 \frac{(1+x^2)\,\ln(1+x)}{(a^2+x^2)(1+a^2x^2)}\,dx = \frac{1}{1+a^2}\left[\int_0^\infty \frac{\ln(1+x)\,dx}{x^2+a^2} - \int_1^\infty \frac{\ln x\,dx}{x^2+a^2}\right].$$

The change of variables $x = at$ and Example 7.4 give

$$\begin{aligned}\int_0^\infty \frac{\ln(1+x)\,dx}{x^2+a^2} &= \frac{1}{a}\int_0^\infty \frac{\ln(1+at)\,dt}{1+t^2} \\ &= \frac{\pi}{4a}\ln(1+a^2) - \frac{1}{a}\int_0^a \frac{\ln t\,dt}{1+t^2}.\end{aligned}$$

Therefore

(7.13)

$$\int_0^1 \frac{(1+x^2)\,\ln(1+x)}{(a^2+x^2)(1+a^2x^2)}\,dx = \frac{1}{1+a^2}\left[\frac{\pi}{4a}\ln(1+a^2) - \frac{1}{a}\int_0^a \frac{\ln x\,dx}{1+x^2} - \int_1^\infty \frac{\ln x\,dx}{x^2+a^2}\right].$$

The change of variables $x = at$ gives

$$\begin{aligned}\int_1^\infty \frac{\ln x\,dx}{x^2+a^2} &= \frac{\ln a}{a}\int_{1/a}^\infty \frac{dt}{1+t^2} + \frac{1}{a}\int_{1/a}^\infty \frac{\ln t\,dt}{1+t^2} \\ &= \frac{\ln a}{a}\int_{1/a}^\infty \frac{dt}{1+t^2} - \frac{1}{a}\int_0^a \frac{\ln u\,du}{1+u^2},\end{aligned}$$

after the change of variables $u = 1/t$ in the last integral. Replacing in (7.13) gives the result.

EXAMPLE 7.6. The last entry of [**6**] discussed here is **4.291.22**

$$\int_0^\infty \frac{x\,\ln(a+x)}{(b^2+x^2)^2}\,dx = \frac{1}{2(a^2+b^2)}\left(\ln b + \frac{\pi a}{2b} + \frac{a^2}{b^2}\ln a\right).$$

As before, start with the identity

$$\frac{x}{(x^2+b^2)^2} = -\frac{d}{dx}\frac{1}{2(x^2+b^2)} \tag{7.14}$$

and integrate by parts to produce

$$\int_0^\infty \frac{x\,\ln(a+x)}{(b^2+x^2)^2}\,dx = \frac{\ln a}{2b^2} + \frac{1}{2}\int_0^\infty \frac{dx}{(x+a)(x^2+b^2)}.$$

This last integral is evaluated by the method of partial fractions to obtain the result.

Summary. The examples presented here, complete the evaluation of every entry in Section 4.291 of the table [**6**]. The entries not appearing here have been presented in [**4, 5, 7**].

8. Integrals yielding partial sums of the zeta function

Some entries of [**6**] contain as the integrand the product of $\ln x$ and a rational function coming from manipulations of a geometric series. This section presents the evaluation of some of these examples. These evaluations can be written in terms of the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n^s} \tag{8.1}$$

and the generalized harmonic numbers

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}. \tag{8.2}$$

EXAMPLE 8.1. Entry **4.231.18** states that

$$\int_0^1 \frac{1-x^{n+1}}{(1-x)^2} \ln x \, dx = -\frac{(n+1)\pi^2}{6} + \sum_{k=1}^{n} \frac{n-k+1}{k^2}. \tag{8.3}$$

This can be expressed as

$$\int_0^1 \frac{1-x^{n+1}}{(1-x)^2} \ln x \, dx = -(n+1)\zeta(2) + (n+1)H_{n,2} - H_{n,1}. \tag{8.4}$$

The evaluation begins with the identity

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k \tag{8.5}$$

and its shift

$$\frac{1-x^{n+1}}{(1-x)^2} = \sum_{k=0}^{n} (k+1)x^k + (n+1) \sum_{k=n+1}^{\infty} x^k. \tag{8.6}$$

Integrate term by term and use the value

$$\int_0^1 x^k \ln x \, dx = -\frac{1}{(k+1)^2} \tag{8.7}$$

to obtain

$$\int_0^1 \frac{1-x^{n+1}}{(1-x)^2} \ln x \, dx = -\sum_{k=0}^{n} \frac{1}{k+1} - (n+1) \sum_{k=n+1}^{\infty} \frac{1}{(k+1)^2}. \tag{8.8}$$

This can now be transformed to the form stated in [**6**].

EXAMPLE 8.2. Entry **4.262.7**

$$\int_0^1 \frac{1-x^{n+1}}{(1-x)^2} (\ln x)^3 \, dx = -\frac{(n+1)\pi^4}{15} + 6\sum_{k=1}^{n} \frac{n-k+1}{k^4} \tag{8.9}$$

is obtained by using (8.6), the identity

$$\int_0^1 (\ln x)^3 x^k \, dx = -\frac{6}{(k+1)^4}, \tag{8.10}$$

and the value

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \zeta(4) = \frac{\pi^4}{90}. \tag{8.11}$$

EXAMPLE 8.3. Replacing x by x^2 is (8.6) gives

$$\frac{1-x^{2n+2}}{(1-x^2)^2} = \sum_{k=0}^{n}(k+1)x^{2k} + (n+1)\sum_{k=n+1}^{\infty} x^{2k}. \tag{8.12}$$

This gives

$$\begin{aligned} \int_0^1 \frac{1-x^{2n+2}}{(1-x^2)^2} \ln x \, dx &= \sum_{k=0}^{n}(k+1)\int_0^1 x^{2k} \ln x \, dx + (n+1)\sum_{k=n+1}^{\infty}\int_0^1 x^{2k} \ln x \, dx \\ &= -\sum_{k=0}^{n} \frac{k+1}{(2k+1)^2} - (n+1)\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)^2}. \end{aligned}$$

The value

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} = \frac{3}{4}\zeta(2) \tag{8.13}$$

is obtained by separating the terms forming the series for $\zeta(2)$ into even and odd indices. Now write

$$\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)^2} = \frac{3}{4}\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{(2k-1)^2} \tag{8.14}$$

to obtain, after some elementary algebraic manipulatons, the evaluation

$$\int_0^1 \frac{1-x^{2n+2}}{(1-x^2)^2} \ln x \, dx = -\frac{3}{4}(n+1)\zeta(2) + \sum_{k=1}^{n} \frac{n-k+1}{(2k-1)^2}. \tag{8.15}$$

This is entry **4.231.16**.

EXAMPLE 8.4. The alternating geometric series

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k \tag{8.16}$$

is used as before to derive the identity

$$\frac{1+(-1)^n x^{n+1}}{(1+x)^2} = (n+1)\sum_{k=0}^{\infty}(-1)^k x^k - \sum_{k=0}^{n}(-1)^k(n-k)x^k. \tag{8.17}$$

Integrating yields

$$\int_0^1 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \ln x \, dx = -(n+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} - \sum_{k=1}^{n} \frac{(-1)^k(n-k+1)}{k^2}. \tag{8.18}$$

This is entry **4.231.17**, written in the form

$$\int_0^1 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \ln x \, dx = -\frac{(n+1)\pi^2}{12} - \sum_{k=1}^{n} \frac{(-1)^k (n-k+1)}{k^2}, \tag{8.19}$$

using the value

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} = \frac{\pi^2}{12}. \tag{8.20}$$

Example 8.5. Entry **4.262.8**

$$\int_0^1 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} (\ln x)^3 \, dx = -\frac{7(n+1)\pi^4}{120} + 6\sum_{k=1}^{n} (-1)^{k-1} \frac{n-k+1}{k^4} \tag{8.21}$$

is obtained by using (8.17) and the identities employed in Example 8.2. The procedure employed in Example 8.3 now gives entry **4.262.9**

$$\int_0^1 \frac{1-x^{2n+2}}{(1-x^2)^2} (\ln x)^3 \, dx = -\frac{(n+1)\pi^4}{16} + 6\sum_{k=1}^{n} \frac{n-k+1}{(2k-1)^4}. \tag{8.22}$$

9. A singular integral

The last evaluation presented here is entry **4.231.10**

$$\int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} = -\frac{\pi^2}{4ab}. \tag{9.1}$$

The parameters a, b have the same sign, so it may be assumed that $a, b > 0$. Observe that this is a singular integral, since the integrand is discontinuous at $x = a/b$.

The change of variables $t = bx/a$ gives

$$\int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} = \frac{1}{ab} \left[\ln \frac{a}{b} \int_0^\infty \frac{dt}{1-t^2} + \int_0^\infty \frac{\ln t \, dt}{1-t^2} \right]. \tag{9.2}$$

The first integral is singular and is computed as the limit as $\varepsilon \to 0$ of

$$\int_0^{1-\varepsilon} \frac{dt}{1-t^2} + \int_{1+\varepsilon}^\infty \frac{dt}{1-t^2} = \frac{1}{2} \ln \left(\frac{2-\varepsilon}{\varepsilon} \right) + \frac{1}{2} \ln \left(\frac{\varepsilon}{2+\varepsilon} \right) = \frac{1}{2} \ln \left(\frac{2-\varepsilon}{2+\varepsilon} \right) \tag{9.3}$$

obtained by the method of partial fraction. Therefore this singular integral has value 0. The second integral is

$$\int_0^\infty \frac{\ln t \, dt}{1-t^2} = 2\int_0^1 \frac{\ln t \, dt}{1-t^2}, \tag{9.4}$$

because the integral over $[1, \infty)$ is the same as over $[0, 1]$. The method of partial fractions and the values

$$\int_0^1 \frac{\ln x \, dx}{1-x} = -\frac{\pi^2}{6} \text{ and } \int_0^1 \frac{\ln x \, dx}{1+x} = -\frac{\pi^2}{12}, \tag{9.5}$$

that appear as entries **4.231.2** and **4.231.1**, respctively, give the final result. These last two entries were evaluated in [**1**].

The change of variables $t = \ln x$ converts this integral into entry **3.417.2**

$$\int_{-\infty}^{\infty} \frac{t\,dt}{a^2e^t - b^2e^{-t}} = \frac{\pi^2}{4ab}. \tag{9.6}$$

The same change of variables gives the evaluation of entry **3.417.1**

$$\int_{-\infty}^{\infty} \frac{t\,dt}{a^2e^t + b^2e^{-t}} = \frac{\pi}{2ab}\ln\frac{b}{a} \tag{9.7}$$

from entry **4.231.8**

$$\int_0^{\infty} \frac{\ln x\,dx}{a^2 + b^2x^2} = -\frac{\pi}{2ab}\ln\frac{b}{a} \tag{9.8}$$

evaluated in [**4**].

Summary. The examples presented here, complete the evaluation of every entry in Section 4.231 of the table [**6**]. The entries not appearing here have been presented in [**4, 5, 7**].

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