ON POLYNOMIALS CONNECTED TO POWERS OF BESSEL FUNCTIONS

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Abstract. The series expansion of a power of the modified Bessel function of the first kind is studied. This expansion involves a family of polynomials introduced by C. Bender et al. New results on these polynomials established here include recurrences in terms of Bell polynomials evaluated at values of the Bessel zeta function. A probabilistic version of an identity of Euler yields additional recurrences. Connections to the umbral formalism on Bessel functions introduced by Cholewinski are established.

1. Introduction

Starting with the power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{a_n} \frac{z^n}{n!}, \quad a_n \neq 0, \tag{1.1}$$

the corresponding expansion for $[f(z)]^r$, $r \in \mathbb{N}$

$$[f(z)]^r = \sum_{n=0}^{\infty} A_n(r) \frac{1}{a_n} \frac{z^n}{n!} \tag{1.2}$$

defines the coefficients $A_n(r)$. For instance, $A_0(r) = 1/a_0^{r-1}$. The identity $[f(z)]^{r+1} = [f(z)]^r \times f(z)$ produces

$$A_n(r+1) = \sum_{j=0}^{n} \binom{n}{j}_{\mathbf{a}} A_j(r), \tag{1.3}$$

where the generalized binomial coefficients $\binom{n}{j}_{\mathbf{a}}$ are defined as

$$\binom{n}{j}_{\mathbf{a}} = \binom{n}{j} \frac{a_n}{a_j a_{n-j}}. \tag{1.4}$$

The example motivating the results presented here comes from work by C. Bender at al [5] on the normalized Bessel function

$$\tilde{I}_\nu(z) = \frac{\nu! 2^\nu}{z^\nu} I_\nu(z) = \sum_{k=0}^{\infty} \frac{\nu!}{k!(k+\nu)!} \left(\frac{z}{2}\right)^{2k+\nu}. \tag{1.5}$$

Here

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!(k+\nu)!} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{1.6}$$

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is the *modified Bessel function of the first kind*. The main result in [5] is stated next.

Theorem 1.1 (Bender et al.)**.** *For $r \in \mathbb{N}$, the power series expansion*

$$\left[\tilde{I}_\nu(z)\right]^r = \sum_{k=0}^{\infty} \frac{\nu!}{k!(k+\nu)!} B_k^{(\nu)}(r) \left(\frac{z}{2}\right)^{2k}, \tag{1.7}$$

holds for polynomials $B_k^{(\nu)}(r)$, determined recursively by

$$B_k^{(\nu)}(r) = r\frac{\nu+k}{\nu+1} B_{k-1}^{(\nu)}(r) + \sum_{j=2}^{k} \frac{b_j(\nu)}{k} \frac{(\nu+1)!}{(\nu+1+j)!} \binom{\nu+k}{j} B_{k-j}^{(\nu)}(r) \tag{1.8}$$

with initial condition $B_0^{(\nu)}(r) = 1$. The sequence $b_j(\nu)$ has the generating function

$$\sum_{k=1}^{\infty} \frac{b_k(\nu)}{(\nu+k)!(k-1)!} x^k = \left(\frac{\sqrt{x}}{\nu+1} \frac{I_\nu(2\sqrt{x})}{I_{\nu+1}(2\sqrt{x})} - 2 \right) \frac{x}{(\nu+1)!}. \tag{1.9}$$

The goal of this work is to present an alternative approach to the expansion (1.7). Section 2 uses the identity (1.3) to derive an expression for $B_k^{(\nu)}(r+1)$ in terms of $B_j^{(\nu)}(r)$. Section 3 identifies $B_k^{(\nu)}(r)$ as Bell polynomials and produces a new recurrence involving the Bessel zeta function. Section 4 uses an identity of Euler to derive a second recurrence for $B_k^{(\nu)}(r)$. Section 5 shows that these polynomials are of binomial type and a link with Cholewinski's theory of Bessel functions is exhibited. The last section is dedicated to arithmetic properties of some sequences related to powers of Bessel functions.

The notation $\nu! = \Gamma(\nu+1)$ is employed throughout.

2. A first identity for $B_n^{(\nu)}(r)$.

The expansion of $\tilde{I}_\nu(z)$, written as

$$\tilde{I}_\nu(\sqrt{z}) = \sum_{k=0}^{\infty} \frac{1}{a_k^{(\nu)}} \frac{z^k}{k!} \tag{2.1}$$

with

$$a_k^{(\nu)} = \frac{(k+\nu)!}{\nu!} 2^{2k}, \tag{2.2}$$

shows that the coefficients $A_k(r)$ in (1.2) are precisely the polynomials $B_k^{(\nu)}(r)$ in (1.7). The fact that $B_n^{(\nu)}(r)$ are polynomials, not a priori clear from their definition, follows from (1.3).

Theorem 2.1. *The functions $B_n^{(\nu)}(r)$ satisfy*

$$B_n^{(\nu)}(r+1) = \sum_{j=0}^{n} \binom{n}{j} \frac{(n+\nu)!\,\nu!}{(\nu+j)!\,(n-j+\nu)!} B_j^{(\nu)}(r). \tag{2.3}$$

Corollary 2.2. *The coefficients $B_n^{(\nu)}(r)$ are polynomials in r of degree n.*

Proof. Proceed by induction on n, writing (2.3) as

$$B_n^{(\nu)}(r+1) - B_n^{(\nu)}(r) = \sum_{j=0}^{n-1} \binom{n}{j} \frac{(n+\nu)!\,\nu!}{(\nu+j)!\,(n-j+\nu)!} B_j^{(\nu)}(r). \tag{2.4}$$

The result follows by differentiating $n+1$ times with respect to r. $\square$

First examples of these polynomials are:

$$B_0^{(\nu)}(r) = 1,\ B_1^{(\nu)}(r) = r,\ B_2^{(\nu)}(r) = \frac{\nu+2}{\nu+1}r^2 - \frac{1}{\nu+1}r$$

$$B_3^{(\nu)}(r) = \frac{(\nu+3)\,(\nu+2)}{(\nu+1)^2}r^3 - 3\frac{\nu+3}{(\nu+1)^2}r^2 + \frac{4}{(\nu+1)^2}r$$

.

Note 2.3. The identity (2.3), established first for $r \in \mathbb{N}$, naturally holds for $r \in \mathbb{R}$. The same principle applies to the recurrences involving $B_n^{(\nu)}(r)$ presented below.

3. An identity for $B_n^{(\nu)}(r)$ in terms of the Bessel zeta function.

This section provides an expression for the polynomials $B_n^{(\nu)}(r)$, starting from

$$\left[\tilde{I}_\nu(z)\right]^r = \exp\left[r\,\log \tilde{I}_\nu(z)\right] \tag{3.1}$$

and using the Hadamard factorization

$$\tilde{I}_\nu(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{j_{\nu,k}^2}\right). \tag{3.2}$$

Here $\{j_{\nu,k}\}_{k\geq 1}$ is the sequence of zeros of $\tilde{J}_\nu(z) = \tilde{I}_\nu(\imath z)$.

The exponential of a power series is computed by

$$\exp\left[\sum_{p=1}^{\infty} a_p\,\frac{z^p}{p!}\right] = \sum_{p=0}^{\infty} \mathfrak{B}_n(a_1,\cdots,a_n)\,\frac{z^n}{n!}, \tag{3.3}$$

where $\mathfrak{B}_n(a_1,\cdots,a_n)$ is the n-th *complete Bell polynomial.* Details appear in [11, p.173]. The first few examples are

$$\mathfrak{B}_0 = 1,\ \mathfrak{B}_1(a_1) = a_1,\ \mathfrak{B}_2(a_1,a_2) = a_1^2 + a_2,\ \mathfrak{B}_3(a_1,a_2,a_3) = a_1^3 + 3a_1a_2 + a_3.$$

From (3.2), it follows that

$$\log \tilde{I}_\nu(z) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p}\zeta_\nu(2p)z^{2p}, \tag{3.4}$$

where $\zeta_\nu(p)$ is the *Bessel zeta function* [10], defined by

$$\zeta_\nu(p) = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^p},\ p > 1. \tag{3.5}$$

A direct application of (3.3) yields the next result.

Theorem 3.1. *Define*

$$a_n = a_n(r) = (-1)^{n-1}(n-1)!\zeta_\nu(2n)r. \tag{3.6}$$

Then $B_n^{(\nu)}(r)$ *is given by*

$$B_n^{(\nu)}(r) = 2^{2n}\frac{(n+\nu)!}{\nu!}\mathfrak{B}_n(a_1(r), \cdots, a_n(r)). \tag{3.7}$$

A recurrence for $B_n^{(\nu)}(r)$ is now obtained from the classical identity [11, p.174]

$$\mathfrak{B}_n(a_1, \cdots, a_n) = \sum_{k=0}^{n-1}\binom{n-1}{k}a_{k+1}\mathfrak{B}_{n-1-k}(a_1, \cdots, a_{n-1-k}) \tag{3.8}$$

for the complete Bell polynomials.

Theorem 3.2. *The polynomials* $B_n^{(\nu)}(r)$ *satisfy the recurrence*

$$B_n^{(\nu)}(r) = r(\nu+n)!\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k\,k!}{(\nu+n-k-1)!}2^{2k+2}\zeta_\nu(2k+2)B_{n-1-k}^{(\nu)}(r).$$

with initial condition $B_0^{(\nu)}(r) = 1$.

Proof. Simply replace (3.7) in (3.8). □

Note 3.3. The recurrence above provides a new proof of Corollary 2.2. However, it is not easy to use, since there is no explicit expression for the coefficients. These involve the values $\zeta_\nu(2k)$, that can be obtained from

$$(n+\nu)\zeta_\nu(2n) = \sum_{r=1}^{n-1}\zeta_\nu(2r)\zeta_\nu(2n-2r), \tag{3.9}$$

established in [8]. The initial condition $\zeta_\nu(2) = \frac{1}{4(\nu+1)}$ shows that $\zeta_\nu(2n)$ is a rational function of ν. These have recently appeared in connection with Narayana polynomials [2]. The first few are

$$\zeta_\nu(4) = \frac{1}{16(\nu+1)^3},\ \zeta_\nu(6) = \frac{1}{16(\nu+1)^4(2\nu+3)},\ \zeta_\nu(8) = \frac{10\nu+11}{256(\nu+1)^6(2\nu^2+7\nu+6)}.$$

A more explicit recurrence for $B_n^{(\nu)}(r)$ is given in the next section.

4. A second recurrence for the polynomials $B_n^{(\nu)}(r)$.

This section describes a new recurrence for the polynomials $B_n^{(\nu)}(r)$. The proof is based on a probabilistic interpretation of a beautiful result of L. Euler, recently used by A. Baricz [4] to discuss properties for powers of Bessel functions.

Theorem 4.1 (Euler)**.** *Let*

$$f(x) = \sum_{n=0}^{\infty}c_n x^n \tag{4.1}$$

and assume $c_0 \neq 0$. *The coefficients* d_n *in*

$$(f(x))^r = \sum_{n=0}^{\infty}d_n x^n \tag{4.2}$$

are given by the recurrence

$$d_n = \frac{1}{c_0}\sum_{k=1}^{n}\left(\frac{k}{n}(r+1)-1\right)c_k d_{n-k},\ n\geq 1 \tag{4.3}$$

with initial condition $d_0 = c_0^r$.

A probabilistic counterpart is stated next. The proof presented here does not require to have a priori knowledge of Euler's formula (4.3).

Lemma 4.2. *Assume* $X_1, X_2, \cdots, X_{r+1}$ *are* $r+1$ *independent identically distributed random variables with vanishing odd moments; that is,*

$$\mathbb{E}X_i^{2k+1} = 0,\ \textit{for all } k\in\mathbb{N}. \tag{4.4}$$

Then

$$\mathbb{E}(X_1+\cdots+X_{r+1})^{2n} = \frac{r+1}{n}\sum_{k=1}^{n}\binom{2n}{2k}k\mathbb{E}X_1^{2k}\mathbb{E}(X_2+\cdots+X_{r+1})^{2n-2k}. \tag{4.5}$$

Proof. Denote $X = X_1$ and $Y = X_2+\cdots+X_{r+1}$ and use the binomial theorem to obtain

$$\begin{aligned}\mathbb{E}X(X+Y)^{2n-1} &= \sum_{k=0}^{2n-1}\binom{2n-1}{k}\mathbb{E}X^{k+1}\mathbb{E}Y^{2n-1-k}\\ &= \sum_{k=0}^{n}\binom{2n-1}{2k-1}\mathbb{E}X^{2k}\mathbb{E}Y^{2n-2k}+\sum_{k=0}^{n-1}\binom{2n-1}{2k}\mathbb{E}X^{2k+1}\mathbb{E}Y^{2n-1-2k}.\end{aligned}$$

The second sum vanishes as it only includes odd moments. Therefore

$$\mathbb{E}X(X+Y)^{2n-1} = \sum_{k=0}^{n}\binom{2n-1}{2k-1}\mathbb{E}X^{2k}\mathbb{E}Y^{2n-2k}.$$

The identity

$$\begin{aligned}\mathbb{E}X(X+Y)^{2n-1} &= \mathbb{E}X_1(X_1+X_2+\cdots+X_{r+1})^{2n-1}\\ &= \mathbb{E}X_2(X_1+X_2+\cdots+X_{r+1})^{2n-1}\\ &= \ldots\\ &= \mathbb{E}X_{r+1}(X_1+X_2+\cdots+X_{r+1})^{2n-1},\end{aligned}$$

comes from the fact that all the random variables have the same distribution. The elementary identity

$$\binom{2n-1}{2k-1} = \frac{k}{n}\binom{2n}{2k} \tag{4.6}$$

yields

$$(r+1)\mathbb{E}X(X+Y)^{2n-1} = \mathbb{E}(X+Y)(X+Y)^{2n-1} = \mathbb{E}(X+Y)^{2n}.$$

This gives the result. □

Theorem 4.3. *The polynomials* $B_n^{(\nu)}(r)$ *satisfy the recurrences*

$$B_n^{(\nu)}(r+1) = \frac{r+1}{n}\sum_{k=1}^{n}k\binom{n}{k}\frac{(\nu+1)_n}{(\nu+1)_k(\nu+1)_{n-k}}B_{n-k}^{(\nu)}(r) \tag{4.7}$$

and

$$B_n^{(\nu)}(r) = \sum_{k=1}^{n} \left[\frac{k(r+1)}{n} - 1 \right] \binom{n}{k} \frac{(\nu+1)_n}{(\nu+1)_k(\nu+1)_{n-k}} B_{n-k}^{(\nu)}(r). \tag{4.8}$$

Proof. Assume $\{X_i\}$ is a collection of independent random variables, identically distributed with a centered beta distribution

$$f_\nu(x) = \begin{cases} \frac{1}{B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}(1-x^2)^{\nu-\frac{1}{2}}, & \text{for } |x|<1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

The associated characteristic function is

$$\varphi_\nu(z) = \mathbb{E}e^{\imath z X_i} = \sum_{n=0}^{\infty} \mathbb{E}X_i^{2n} \frac{(-1)^n z^{2n}}{(2n)!} = \tilde{J}_\nu(z) \tag{4.10}$$

in view of [1, Entry 9.1.20], [9, Entry 8.411.8]

$$J_\nu(z) = \frac{(z/2)^\nu}{\pi^{1/2}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \cos(zt)\, dt.$$

In particular

$$\mathbb{E}X_i^{2n} = \frac{1}{(\nu+1)_n} \frac{(2n)!}{n!2^{2n}} \text{ and } \mathbb{E}X_i^{2n+1} = 0. \tag{4.11}$$

The definition of $B_n^{(\nu)}(r)$ and

$$\left[\tilde{J}_\nu(z)\right]^r = \left(\mathbb{E}e^{\imath z X_1}\right)^r = \mathbb{E}e^{\imath z(X_1+\cdots+X_r)} \tag{4.12}$$

imply

$$\mathbb{E}(X_1+\cdots+X_r)^{2n} = \frac{1}{(\nu+1)_n}\frac{(2n)!}{n!2^{2n}} B_n^{(\nu)}(r). \tag{4.13}$$

Replacing in (4.5) yields

$$\frac{1}{(\nu+1)_n}\frac{(2n)!}{n!2^{2n}} B_n^{(\nu)}(r+1) = \frac{r+1}{n}\sum_{k=1}^{n}\binom{2n}{2k} k \frac{1}{(\nu+1)_k}\frac{(2k)!}{k!2^{2k}} \frac{1}{(\nu+1)_{n-k}} \frac{(2n-2k)!}{(n-k)!2^{2n-2k}} B_{n-k}^{(\nu)}(r),$$

that reduces to (4.7).

The second recurrence is obtained by applying the binomial formula

$$\begin{aligned} \mathbb{E}(X_1+\cdots+X_{r+1})^{2n} &= \sum_{k=0}^{n}\binom{2n}{2k}\mathbb{E}X_1^{2k}\mathbb{E}(X_2+\cdots+X_{r+1})^{2n-2k} \\ &= \mathbb{E}(X_2+\cdots+X_{r+1})^{2n} + \sum_{k=1}^{n}\binom{2n}{2k}\mathbb{E}X_1^{2k}\mathbb{E}(X_2+\cdots+X_{r+1})^{2n-2k}, \end{aligned}$$

to obtain

$$\mathbb{E}(X_2+\cdots+X_{r+1})^{2n} = \mathbb{E}(X_1+\cdots+X_{r+1})^{2n} - \sum_{k=1}^{n}\binom{2n}{2k}\mathbb{E}X_1^{2k}\mathbb{E}(X_2+\cdots+X_{r+1})^{2n-2k}.$$

In terms of the polynomials $B_n^{(\nu)}(r)$, the previous relation becomes

$$\begin{aligned}\frac{1}{(\nu+1)_n}\frac{(2n)!}{n!2^{2n}}B_n^{(\nu)}(r) &= \frac{1}{(\nu+1)_n}\frac{(2n)!}{n!2^{2n}}B_n^{(\nu)}(r+1) \\ &\quad -\sum_{k=1}^{n}\binom{2n}{2k}\frac{1}{(\nu+1)_k}\frac{(2k)!}{k!2^{2k}}\frac{1}{(\nu+1)_{n-k}}\frac{(2n-2k)!}{(n-k)!2^{2n-2k}}B_{n-k}^{(\nu)}(r) \\ &= \frac{1}{(\nu+1)_n}\frac{(2n)!}{n!2^{2n}}\left(B_n^{(\nu)}(r+1)-\sum_{k=1}^{n}\binom{n}{k}\frac{(\nu+1)_n}{(\nu+1)_k(\nu+1)_{n-k}}B_{n-k}^{(\nu)}(r)\right).\end{aligned}$$

Using the expression for $B_n^{(\nu)}(r)$ from (4.7) yields (4.8). □

Note 4.4. The identity (2.3) follows by replacing (4.8) into (4.7).

5. The Cholewinski connection

The normalized Bessel function (1.5) is written as

$$\tilde{I}_\nu(z) = \sum_{n=0}^{\infty}\frac{z^{2n}}{b_{2n}(\nu)} \tag{5.1}$$

where

$$b_{2n}(\nu) = 2^{2n}\frac{n!(n+\nu)!}{\nu!}. \tag{5.2}$$

In his treatise [7], Cholewinski defines a modified binomial coefficient

$$\binom{b_{2n}(\nu)}{b_{2k}(\nu)} = \frac{b_{2n}(\nu)}{b_{2n-2k}(\nu)\,b_{2k}(\nu)} = \binom{n}{k}\frac{\nu!\,(\nu+n)!}{(\nu+n-k)!\,(\nu+k)!} \tag{5.3}$$

and a new convolution by

$$(x*y)^\alpha = \sum_{k=0}^{+\infty}\binom{b_\alpha(\nu)}{b_{2k}(\nu)}x^{2k}y^{\alpha-2k} = y^\alpha\,{}_2F_1\left(\left.\begin{matrix}-\frac{\alpha}{2},-\frac{\alpha}{2}-\nu\\ \nu+1\end{matrix}\right|\frac{x^2}{y^2}\right). \tag{5.4}$$

Here ${}_2F_1$ is the classical hypergeometric function.

In this notation

$$\left[\tilde{I}_\nu(z)\right]^r = \sum_{n=0}^{\infty}B_n^{(\nu)}(r)\frac{z^{2n}}{b_{2n}}. \tag{5.5}$$

The next result gives a recurrence for a normalization of the polynomials $B_n^{(\nu)}(r)$ in Cholewinski's notations.

Theorem 5.1. *The sequence of normalized polynomials*

$$\tilde{B}_n^{(\nu)}(r) = \frac{(2n)!}{n!2^{2n}(\nu+1)_n}B_n^{(\nu)}(r) \tag{5.6}$$

is of binomial type; that is, they satisfy

$$\tilde{B}_n^{(\nu)}(r+s) = \sum_{k=0}^{n}\binom{n}{k}\tilde{B}_k^{(\nu)}(r)\tilde{B}_{n-k}^{(\nu)}(s). \tag{5.7}$$

The (unnormalized) polynomials $B_n^{(\nu)}(r)$ satisfy the identity

$$B_n^{(\nu)}(r+s) = \sum_{k=0}^{n}\binom{b_{2n}}{b_{2k}}B_k^{(\nu)}(r)B_{n-k}^{(\nu)}(s). \tag{5.8}$$

Proof. Start with

$$\begin{aligned}\left[\tilde{I}_{\nu}(z)\right]^{r+s} &= \left[\tilde{I}_{\nu}(z)\right]^{r}\left[\tilde{I}_{\nu}(z)\right]^{s} \\ &= \sum_{p,q} B_{p}^{(\nu)}(r)\frac{z^{2p}}{b_{2p}} B_{q}^{(\nu)}(s)\frac{z^{2q}}{b_{2q}} \\ &= \sum_{n=0}^{\infty}\left(\sum_{p=0}^{n}\frac{b_{2n}}{b_{2p}b_{2n-2p}} B_{p}^{(\nu)}(r) B_{n-p}^{(\nu)}(s)\right)\frac{z^{2n}}{b_{2n}}.\end{aligned}$$

The result now follows from (5.5).

An alternative proof of (5.6) uses the moment representation

$$B_{n}^{(\nu)}(r) = (\nu+1)_{n}\frac{n!}{2n!}2^{2n}\mathbb{E}\left(X_{1}+\cdots+X_{r}\right)^{2n} \tag{5.9}$$

where $\{X_i\}$ are independent, identically distributed random variables with probability density (4.9). Simply observe that

$$\begin{aligned}B_{n}^{(\nu)}(r+s) &= (\nu+1)_{n}\frac{n!}{(2n)!}2^{2n}\mathbb{E}\left(X_{1}+\cdots+X_{r}+X_{r+1}+\cdots+X_{r+s}\right)^{2n} \\ &= (\nu+1)_{n}\frac{n!}{(2n)!}2^{2n}\mathbb{E}\left(\left(X_{1}+\cdots+X_{r}\right)+\left(Y_{1}+\cdots+Y_{s}\right)\right)^{2n}\end{aligned}$$

with $Y_j = X_{r+j}$ and use the binomial theorem. □

Theorem 5.2. *The polynomials $B_{n}^{(\nu)}(r)$ are given by*

$$B_{n}^{(\nu)}(r) = \frac{\mathbb{E}(X_{1}+\cdots+X_{r})^{2n}}{\mathbb{E}(X_{1})^{2n}}. \tag{5.10}$$

The normalized polynomials are simply given by

$$\tilde{B}_{n}^{(\nu)}(r) = \mathbb{E}(X_{1}+\cdots+X_{r})^{2n}. \tag{5.11}$$

Proof. The moment representation

$$B_{n}^{(\nu)}(r) = (\nu+1)_{n}\frac{n!}{(2n)!}2^{2n}\mathbb{E}(X_{1}+\cdots+X_{r})^{2n} \tag{5.12}$$

is simplified using

$$B_{n}^{(\nu)}(1) = 1 = (\nu+1)_{n}\frac{n!}{(2n)!}2^{2n}\mathbb{E}(X_{1})^{2n}. \tag{5.13}$$

□

Note 5.3. The integrals

$$W_{n}(s) = \int_{[0,1]^{n}}\left|\sum_{k=1}^{n} e^{2\pi\imath x_{k}}\right|^{s} d\mathbf{x} \tag{5.14}$$

have recently appeared in [6] in the study of short random walks in the plane. Their generating function is given by

$$\sum_{s=0}^{\infty} W_{n}(2s)\frac{(-z)^{s}}{s!s!} = \left[J_{0}(2\sqrt{z})\right]^{n}. \tag{5.15}$$

The relation

$$J_{0}(2\sqrt{z}) = \tilde{J}_{0}(2\sqrt{z}) \tag{5.16}$$

and

$$\left[\tilde{J}_0(2\sqrt{z})\right]^n = \sum_{s=0}^{\infty} B_s^{(0)}(n) \frac{(\imath\sqrt{z})^{2s}}{s!s!}, \tag{5.17}$$

give the link between the integrals $W_n(2s)$ and the polynomials $B_n^{\nu}(r)$ as

$$W_n(2s) = B_s^{(0)}(n). \tag{5.18}$$

6. Some integer sequences

The sequences $\{a_k\}$ and $\{M_k\}$ are defined there in [5] by the generating function

$$x\left(\frac{\sqrt{x}I_0(2\sqrt{x})}{I_1(2\sqrt{x})} - 2\right) = \sum_{k=1}^{\infty} a_k x^k \tag{6.1}$$

and

$$M_k = (-1)^{k+1}(k+1)!k!a_{k+1}. \tag{6.2}$$

In particular, it is stated that $\{M_k\}$ is an integer sequence. This is proved next. An expression for M_k in terms of the Bessel zeta function is given first.

Theorem 6.1. *The coefficients M_k are given by*

$$M_k = k!(k+1)!2^{2k}\zeta_1(2k). \tag{6.3}$$

Proof. The identity

$$\frac{d}{dz}I_{\nu+1}(z) = I_\nu(z) - \frac{\nu+1}{z}I_{\nu+1}(z), \tag{6.4}$$

implies

$$\frac{d}{dz}\log I_{\nu+1}(z) = \frac{I_\nu(z)}{I_{\nu+1}(z)} - \frac{\nu+1}{z}. \tag{6.5}$$

Then (3.4) gives

$$\frac{d}{dz}\log I_{\nu+1}(z) = \frac{\nu+1}{z} + 2\sum_{p=1}^{\infty}(-1)^{p+1}\zeta_{\nu+1}(2p)z^{2p-1}, \tag{6.6}$$

and (6.4) then produces

$$\frac{I_\nu(2\sqrt{x})}{I_{\nu+1}(2\sqrt{x})} = \frac{\nu+1}{\sqrt{x}} + 2\sum_{p=1}^{\infty}(-1)^{p+1}\zeta_{\nu+1}(2p)(2\sqrt{x})^{2p-1}. \tag{6.7}$$

Therefore

$$x\left(\frac{\sqrt{x}I_\nu(2\sqrt{x})}{(\nu+1)I_{\nu+1}(2\sqrt{x})} - 2\right) = -x + \frac{1}{\nu+1}\sum_{\ell=1}^{\infty}(-1)^{\ell+1}\zeta_{\nu+1}(2\ell)2^{2\ell}x^{\ell+1}. \tag{6.8}$$

The special case $\nu = 0$ gives the result. □

Corollary 6.2. *The number M_k is an integer.*

Proof. The sequence

$$\tilde{a}_n = 2^{2n+1}(n+1)!(n-1)!\zeta_1(2n) \tag{6.9}$$

has been shown to be an integer sequence in [3]. The identity

$$M_n = \frac{n\tilde{a}_n}{2} \tag{6.10}$$

shows that $M_n \in \mathbb{N}$. Indeed, this is clear if $\tilde{a}_n$ is even. For $\tilde{a}_n$ odd, it must be that $n = 2(2^m - 1)$ (see Theorem 8.1 in [3]) and the result follows. □

Note 6.3. The recurrence

$$M_n = \sum_{r=1}^{n-1} \frac{\binom{n}{r}^2}{(r+1)(n-r+1)} M_r M_{n-r} \tag{6.11}$$

with initial conditions $M_1 = M_2 = 1$ follows from the recurrence for $\zeta_1(2n)$ given in [3].

At the end of [5], the authors define the more general sequence $b_n(\nu)$, that has already appeared in (1.8), by the generating function

$$\sum_{n=1}^{\infty} \frac{x^n}{(\nu+n)!(n-1)!} b_n(\nu) = \frac{x}{(\nu+1)!} \left(\frac{\sqrt{x}}{\nu+1} \frac{I_\nu(2\sqrt{x})}{I_{\nu+1}(2\sqrt{x})} - 2 \right), \tag{6.12}$$

and it is stated that $b_n(\nu)$ is an integer sequence. This is incorrect, since it is easy to check that

$$b_2(\nu) = \frac{1}{\nu+1}. \tag{6.13}$$

The next result presents a possible modification.

Theorem 6.4. *The sequence $b_n(\nu)$ defined by (6.12) is given by*

$$b_n(\nu) = \frac{(n+\nu)!(n-1)!}{(\nu+1)!(\nu+1)} (-1)^n 2^{2n-2} \zeta_{\nu+1}(2n-2), \text{ for } n \geq 2. \tag{6.14}$$

For $\nu \in \mathbb{N}$, the modified sequence

$$\tilde{b}_n(\nu) = (-1)^n b_n(\nu) \frac{(\nu+1)!(\nu+1)}{(\nu+n)!(n-1)!} \prod_{k=1}^{n} (k+\nu)^{\lfloor n/k \rfloor}, \text{ for } n \geq 2 \tag{6.15}$$

takes integer values.

Proof. Comparing (6.8) and (6.12) yields

$$b_n(\nu) = \frac{(\nu+n)!(n-1)!}{(\nu+1)!(\nu+1)} (-1)^n \zeta_{\nu+1}(2n-2). \tag{6.16}$$

The function

$$\phi_{2n}(\nu) = 2^{2n} \zeta_\nu(2n) \prod_{k=1}^{n} (\nu+k)^{\lfloor n/k \rfloor} \tag{6.17}$$

has been shown to be a polynomial [10] of degree $1 - 2n + \sum_{k=1}^{n} \lfloor n/k \rfloor$, with positive integer coefficients. This is the *Rayleigh polynomial*. Then

$$b_\nu(n) = \frac{(n+\nu)!(n-1)!(-1)^n \phi_{2n-2}(\nu+1)}{(\nu+1)!(\nu+1) \prod_{k=1}^{n-1} (k+\nu+1)^{\lfloor (n-1)/k \rfloor}}, \tag{6.18}$$

and the modified sequence (6.15) reduces to $\phi_{2n-2}(\nu+1)$, giving the result. □

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