A SYMBOLIC APPROACH TO SOME IDENTITIES FOR BERNOULLI-BARNES POLYNOMIALS

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Abstract. A symbolic method is used to establish some properties of the Bernoulli-Barnes polynomials.

1. Introduction

The Bernoulli numbers B_n, defined by their exponential generating function

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \tag{1.1}$$

have produced a variety of generalizations in the literature. The so-called Bernoulli-Barnes numbers $B_k(\mathbf{a})$, defined by

$$\prod_{j=1}^{n} \frac{z}{e^{a_j z}-1} = \sum_{k=0}^{\infty} \frac{B_k(\mathbf{a})}{k!} z^k, \tag{1.2}$$

depend on a multi-dimensional parameter $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{C}^n$. The Bernoulli numbers correspond to $n = 1$ and $\mathbf{a} = 1$.

For any sequence of numbers $\{a_j\}$ with exponential generation function $f(z) = \sum_{j=0}^{\infty} a_j \frac{z^j}{j!}$, associate the sequence of polynomials $A_j(x) = \sum_{\ell=0}^{j} \binom{j}{\ell} a_{j-\ell} x^{\ell}$. An elementary argument shows that $e^{xz} f(z)$ is the exponential generating function for $\{A_j(x)\}$. This produces, from $B_k(\mathbf{a})$, the *Bernoulli-Barnes polynomials*

$$B_j(x;\mathbf{a}) = \sum_{\ell=0}^{j} \binom{j}{\ell} B_{j-\ell}(\mathbf{a}) x^{\ell} \tag{1.3}$$

with exponential generating function

$$\sum_{j=0}^{\infty} B_j(x;\mathbf{a}) \frac{z^j}{j!} = e^{xz} \prod_{k=1}^{n} \frac{z}{e^{a_k z}-1}. \tag{1.4}$$

In the special case $\mathbf{1} = (1, \cdots, 1)$ one obtains the Nörlund polynomials $B_j(x;\mathbf{1})$

$$\sum_{j=0}^{\infty} B_j(x;\mathbf{1}) \frac{z^j}{j!} = e^{xz} \frac{z^n}{(e^z-1)^n}. \tag{1.5}$$

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The Bernoulli-Barnes numbers $B_k(\mathbf{a})$ can be expressed in terms of the Bernoulli numbers B_k by the multiple sum

$$B_k(\mathbf{a}) = \sum_{m_1+\cdots+m_n=k} \binom{k}{m_1,\cdots,m_n} a_1^{m_1-1}\cdots a_n^{m_n-1} B_{m_1}\cdots B_{m_n}. \tag{1.6}$$

Therefore $a_1\cdots a_n B_k(\mathbf{a})$ is also a polynomial in $\mathbf{a}$. Some parts of the literature refer to them as the Bernoulli-Barnes polynomials. The reader should be aware of this share of nomenclature.

The first result requires the notion of a self-dual sequence. Recall that $\{a_n\}$ is called self-dual if it satisfies

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k, \qquad \text{for all } n \in \mathbb{N}. \tag{1.7}$$

The recent study [1] contains the following statement as Corollary 5.4:

Let $\mathbf{a} = (a_1,\cdots,a_n) \in \mathbb{C}^n$ *with* $A = a_1+\cdots+a_n \neq 0$. *Then the sequence* $\{(-1)^n A^{-n} B_n(\mathbf{a}) : n \in \mathbb{Z}_{\geq 0}\}$ *is a self-dual sequence.*

The authors state that

It would be interesting to prove this statement directly.

Section 5 describes self-dual sequences and provides the requested direct proof.

The arguments presented here are in the spirit of symbolic calculus. In this framework, one defines a *Bernoulli symbol* $\mathcal{B}$ and an *evaluation map eval* such that

$$eval\,(\mathcal{B}^n) = B_n. \tag{1.8}$$

The reader is referred to [2] and [3] for the rules of this method. To illustrate the main idea, and omitting the *eval* operator to simplify notation, consider the symbolic identity

$$e^{\mathcal{B}z} = \frac{z}{e^z-1}. \tag{1.9}$$

This is explained by the identities

$$eval\left(e^{\mathcal{B}z}\right) = eval\left(\sum_{n=0}^{\infty} \frac{\mathcal{B}^n}{n!} z^n\right) = \sum_{n=0}^{\infty} \frac{eval\,(\mathcal{B}^n)}{n!} z^n = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!} = \frac{z}{e^z-1}. \tag{1.10}$$

The symbolic version of the Bernoulli polynomials $B_n(x)$, defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{z e^{xz}}{e^z-1} \tag{1.11}$$

is simply (where the *eval* map has been omitted again)

$$B_n(x) = (\mathcal{B}+x)^n. \tag{1.12}$$

The principle of symbolic calculus is to perform all computations replacing the Bernoulli polynomial $B_n(x)$ by the symbol $(\mathcal{B}+x)^n$ and, at the end of the process, apply the evaluation map to obtain the result. The basic expression for Bernoulli polynomials in terms of Bernoulli numbers illustrates the method:

$$B_n(x) = (\mathcal{B}+x)^n = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}^k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{1.13}$$

The symbolic representation of the Bernoulli-Barnes numbers is obtained from a collection of n independent Bernoulli symbols $\{\mathcal{B}_i\}_{1 \leq i \leq n}$, where independence is understood in the sense that

$$e^{z(\mathcal{B}_i+\mathcal{B}_j)} = e^{z\mathcal{B}_i} e^{z\mathcal{B}_j}, \text{ for any } i \neq j. \tag{1.14}$$

Then the Bernoulli-Barnes numbers $B_k(\mathbf{a})$ are given in terms $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathcal{B} = (\mathcal{B}_1, \cdots, \mathcal{B}_n)$ by

$$B_k(\mathbf{a}) = \frac{1}{|\mathbf{a}|} (\mathbf{a} \cdot \mathcal{B})^k \tag{1.15}$$

where

$$\mathbf{a} \cdot \mathcal{B} = \sum_{k=1}^{n} a_k \mathcal{B}_k \text{ and } |\mathbf{a}| = \prod_{k=1}^{n} a_k. \tag{1.16}$$

Similarly, the Bernoulli-Barnes polynomials are represented symbolically by

$$B_k(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|} (x + \mathbf{a} \cdot \mathcal{B})^k. \tag{1.17}$$

2. A DIFFERENCE FORMULA

The section in [1] containing the requested proof begins with a difference formula for the Bernoulli-Barnes polynomials. A direct proof by symbolic arguments is presented here. For any $L \subset \{1, \cdots, n\}$, say $L = \{i_1, \cdots, i_r\}$, introduce the notation

$$\mathbf{a}_L = (a_{i_1}, \cdots, a_{i_r}). \tag{2.1}$$

In general, any symbol with a set $L \subset \{1, \cdots, n\}$ as a subscript, indicates that the indices appearing in the symbol should be restricted to those in the set L. For instance, $\mathbf{a}_{\{2,5\}} = (a_2, a_5)$ and $|\mathbf{a}|_{\{2,5\}} = a_2 a_5$.

Theorem 5.1 in [1] is restated here.

Theorem 2.1. *For* $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{C}^n$ *and* $A = \sum_{i=1}^{n} a_i$, *we have the difference formula*

$$(-1)^m B_m(-x; \mathbf{a}) - B_m(x; \mathbf{a}) = m! \sum_{\ell=0}^{n-1} \sum_{|L|=\ell} \frac{B_{m-n+\ell}(x; \mathbf{a}_L)}{(m-n+\ell)!} \tag{2.2}$$

with $B_m(x; \mathbf{a}_L) = x^m$ *if* $L = \emptyset$. *Furthermore,*

$$B_m(x + A; \mathbf{a}) = (-1)^m B_m(-x; \mathbf{a}). \tag{2.3}$$

It is shown that Theorem 2.1 is a special case of a general expansion formula. A variety of proofs are presented below. The conditions imposed on the function f in the statement of Theorem 2.2 are those required for the existence of the expressions appearing in it. Those functions will be called *reasonable*. In particular polynomials are reasonable functions. Here $f^{(j)}(x)$ represents the j-th derivative of f.

Theorem 2.2. *Let f be a reasonable function. Then, with* $\mathbf{a} = (a_1, \cdots, a_n)$,

$$f(x - \mathbf{a} \cdot \mathcal{B}) = \sum_{j=0}^{n} \sum_{|J|=j} |a|_{J^*} f^{(n-j)} \left(x + (\mathbf{a} \cdot \mathcal{B})_J\right) \tag{2.4}$$

where $J \subset \{1, \cdots, n\}$ *and* $J^* = \{1, \cdots, n\} \setminus J$. *Moreover,*

$$f\left(x + A + \mathbf{a} \cdot \mathcal{B}\right) = f\left(x - \mathbf{a} \cdot \mathcal{B}\right). \tag{2.5}$$

Example 2.3. The theorem gives, for $n = 2$ and any reasonable function f, the relation

$$\begin{aligned} f(x - a_1\mathcal{B}_1 - a_2\mathcal{B}_2) &= f(x + a_1\mathcal{B}_1 + a_2\mathcal{B}_2) \\ &+ a_1 f'(x + a_2\mathcal{B}_2) + a_2 f'(x + a_1\mathcal{B}_1) + a_1 a_2 f''(x). \end{aligned}$$

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Note 2.4. The classical differentiation formula

$$\left(\frac{d}{dx}\right)^j \frac{B_n(x)}{n!} = \frac{B_{n-j}(x)}{(n-j)!} \tag{2.6}$$

shows that Theorem 2.1 is the special case $f(x) = x^m/m!$ of Theorem 2.2.

The proof of Theorem 2.2 uses some basic identities of symbolic calculus. The proofs are presented here for completeness.

Lemma 2.5. *Let g be a reasonable function. Then*

$$g(-\mathcal{B}) = g(\mathcal{B} + 1) = g(\mathcal{B}) + g'(0). \tag{2.7}$$

In particular,

$$-\mathcal{B} = \mathcal{B} + 1. \tag{2.8}$$

Proof. The proof is presented for the monomial $g(x) = x^k$, the general case follows by linearity. The exponential generating function of $(-\mathcal{B})^k$ is

$$\begin{aligned} \sum_{k \geq 0} \frac{(-\mathcal{B})^k z^k}{k!} &= \exp(-\mathcal{B}z) = \frac{-z}{e^{-z} - 1} = \frac{ze^z}{e^z - 1} \\ &= e^z e^{\mathcal{B}z} = e^{z(\mathcal{B}+1)} \\ &= \sum_{k \geq 0} \frac{(\mathcal{B}+1)^k z^k}{k!}, \end{aligned}$$

which proves the first identity. Now since $g(x) = x^k$ produces $g'(0) = \delta_{k-1}$ (the Kronecker delta), it follows that

$$\sum_{k \geq 0} \frac{\mathcal{B}^k z^k}{k!} + \sum_{k \geq 0} \delta_{k-1} \frac{z^k}{k!} = \frac{z}{e^z - 1} + z = \frac{ze^z}{e^z - 1} = e^{(\mathcal{B}+1)z} = \sum_{k \geq 0} \frac{(\mathcal{B}+1)^k z^k}{k!} \tag{2.9}$$

proving the second identity. □

The first proof of Theorem 2.2 is given next.

Proof. Lemma 2.5 applied to $g(\mathcal{B}) = f(x + a\mathcal{B})$ gives

$$f(x - a\mathcal{B}) = f(x + a\mathcal{B}) + af'(x). \tag{2.10}$$

This is the result for $n = 1$. The general case is obtained by a direct induction argument. □

3. An operational Calculus proof

This section presents a proof of Theorem 2.2 based on the action of the operator T_a on a function f by

$$T_a[f(x)] = f(x - a\mathcal{B}). \tag{3.1}$$

Naturally

$$T_{a_1} \circ T_{a_2}[f(x)] = T_{a_1}[f(x - a_1\mathcal{B}_1)] = f(x - a_1\mathcal{B}_1 - a_2\mathcal{B}_2) \tag{3.2}$$

showing that T_{a_1} and T_{a_2} commute with each other. On the other hand, since $f(x - \mathbf{a}\cdot\mathcal{B}) = f(x + \mathbf{a}\cdot\mathcal{B}) + af'(x)$, the operator T_a can be formally expressed as

$$T_a = e^{a\mathcal{B}\frac{\partial}{\partial x}} + a\frac{\partial}{\partial x}, \tag{3.3}$$

so that T_a is the sum of two commuting operators. The composition rule

$$\begin{aligned} T_{a_1} \circ T_{a_2} &= e^{(a_1\mathcal{B}_1 + a_2\mathcal{B}_2)\frac{\partial}{\partial x}} \\ &+ a_1\frac{\partial}{\partial x}e^{a_2\mathcal{B}_2\frac{\partial}{\partial x}} + a_2\frac{\partial}{\partial x}e^{a_1\mathcal{B}_1\frac{\partial}{\partial x}} \\ &+ a_1a_2\frac{\partial^2}{\partial x^2} \end{aligned} \tag{3.4}$$

gives the result of Theorem 2.2 for $n = 2$. The general case follows from the identity

$$\begin{aligned} T_{a_1} \circ \cdots \circ T_{a_n} &= \prod_{j=1}^{n}\left(e^{a_j\mathcal{B}_j\frac{\partial}{\partial x}} + a_j\frac{\partial}{\partial x}\right) \\ &= \sum_{j=0}^{n}\sum_{|J|=j}|\mathbf{a}|_{J^*}\frac{\partial^{n-j}}{\partial x^{n-j}}e^{(\mathbf{a}\cdot\mathcal{B})_J\frac{\partial}{\partial x}}. \end{aligned} \tag{3.5}$$

4. A new symbol and another proof

This section provides a proof of Theorem 2.2 based on the uniform symbol $\mathcal{U}$ defined by the relation

$$f(x + \mathcal{U}) = \int_0^1 f(x + u)\,du. \tag{4.1}$$

The uniform symbol acts like the inverse of the Bernoulli symbol, in a sense made precise in the next statement.

Proposition 4.1. *Let $\mathcal{B}$ and $\mathcal{U}$ be the Bernoulli and uniform symbols, respectively. Then, for any reasonable function f,*

$$f(x + \mathcal{U} + \mathcal{B}) = f(x). \tag{4.2}$$

In particular, the relations

$$g(x + \mathcal{B}) = h(x) \text{ and } h(x + \mathcal{U}) = g(x) \tag{4.3}$$

are equivalent.

Proof. The generating function

$$\sum_{n\geq 0}\frac{(x + \mathcal{U} + \mathcal{B})^n}{n!}z^n = e^{zx + z\mathcal{U} + z\mathcal{B}} = e^{zx}\frac{z}{e^z - 1}\frac{e^z - 1}{z} = e^{zx} \tag{4.4}$$

shows that $(z + \mathcal{U} + \mathcal{B})^n = z^n$. The result extends to a general function g by linearity. $\square$

An interpretation of the special case $n = 1$ in Theorem 2.2 is provided next. This is

$$f(x - a\mathcal{B}) = f(x + a\mathcal{B}) + af'(x). \tag{4.5}$$

Now replace x by $x + a\mathcal{U}$ and use the relation $-\mathcal{B} = \mathcal{B} + 1$ to convert the left-hand side of (4.5) to

$$f(x - a\mathcal{B} + a\mathcal{U}) = f(x + a(\mathcal{B} + 1) + a\mathcal{U}) = f(x + a). \tag{4.6}$$

The right-hand side of (4.5) becomes

$$f(x + a\mathcal{B} + a\mathcal{U}) + af'(x + a\mathcal{U}) = f(x) + af'(x + a\mathcal{U}). \tag{4.7}$$

It follows that Theorem 2.2, in the case $n = 1$, is equivalent to the fundamental theorem of Calculus

$$f(x + a) = f(x) + \int_0^a f'(x + u)\, du. \tag{4.8}$$

This is now written in the form

$$\Delta_a f(x) = af'(x + a\mathcal{U}), \tag{4.9}$$

where Δ_a is the forward difference operator with step size a.

The proof of Theorem 2.2 for arbitrary n follows from the method above and the elementary identity

$$\prod_{i=1}^{n} \Delta_{a_i} f(x) = a_1 \cdots a_n f^{(n)}(x + a_1\mathcal{U}_1 + \cdots + a_n\mathcal{U}_n). \tag{4.10}$$

5. Self-duality property for the Bernoulli-Barnes polynomials

Given a sequence $\{a_k\}$ define a new sequence $\{a_k^*\}$ by the rule

$$a_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k. \tag{5.1}$$

The inversion formula [4, p. 192] gives

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k^*. \tag{5.2}$$

The sequence $\{a_n^*\}$ is called the dual of $\{a_n\}$. A sequence is called *self-dual* if it agrees with its dual. Examples of self-dual sequences have been discussed in [6, 7]. For example, the fact that the sequence $\{(-1)^n B_n\}$ is self-dual is equivalent to the classical identity

$$(-1)^n B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \tag{5.3}$$

which, expressed symbolically, is nothing but (2.8). In [1] the authors prove the next result as Corollary 5.. This is an extension of (5.3) to the Bernoulli-Barnes polynomials and ask for a more direct proof. Such a proof is presented next.

Theorem 5.1. *Let* $\mathbf{a} = (a_1, \cdots, a_n)$ *and* $A = a_1 + \cdots + a_n \neq 0$. *Then the sequence*

$$p_n = (-1)^n A^{-n} B_n(\mathbf{a}) \tag{5.4}$$

is self-dual.

Proof. Observe that

$$\begin{aligned}
p_n^* &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k p_k \\
&= \sum_{k=0}^{n} \binom{n}{k} A^{-k} (\mathbf{a} \cdot \mathcal{B})^k \\
&= \left(1 + \frac{1}{A} \mathbf{a} \cdot \mathcal{B} \right)^n \\
&= A^{-n} (A + \mathbf{a} \cdot \mathcal{B})^n \\
&= A^{-n} \left(a_1(1 + \mathcal{B}_1) + \cdots + a_n(1 + \mathcal{B}_n) \right)^n \\
&= A^{-n} (-\mathbf{a} \cdot \mathcal{B})^n \\
&= (-1)^n A^{-n} B_n(\mathbf{a}) \\
&= p_n.
\end{aligned}$$

This completes the proof. $\square$

The authors of [1] then ask for a direct proof of the following symmetry formula. Such a proof is presented next.

Theorem 5.2. *Let* $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$ *with* $A = \sum_{k=1}^{n} a_k \neq 0$. *Then for any integers* l, $m \geq 0$,

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} A^{m-k} B_{l+k}(x; \mathbf{a}) = (-1)^l \sum_{k=0}^{l} \binom{l}{k} A^{l-k} B_{m+k}(-x; \mathbf{a}), \tag{5.5}$$

and

$$\begin{aligned}
&\frac{(-1)^m}{m+l+2} \sum_{k=0}^{m} \binom{m+1}{k} (l+k+1) A^{m+1-k} B_{l+k}(x; \mathbf{a}) + \\
&\qquad \frac{(-1)^l}{m+l+2} \sum_{k=0}^{l} \binom{l+1}{k} (m+k+1) A^{l+1-k} B_{m+k}(x; \mathbf{a}) = \\
&\qquad\qquad (-1)^{m+1} B_{l+m+1}(x; \mathbf{a}) + (-1)^{l+1} B_{l+m+1}(-x; \mathbf{a}).
\end{aligned} \tag{5.6}$$

Proof. The left-hand side of (5.5) can be written as

$$\begin{aligned}
(-1)^m \sum_{k=0}^{m} \binom{m}{k} A^{m-k} B_{l+k}(x; \mathbf{a}) &= (-1)^m \sum_{k=0}^{m} \binom{m}{k} A^{m-k} (x + \mathbf{a} \cdot \mathcal{B})^{l+k} \\
&= (-1)^m (x + \mathbf{a} \cdot \mathcal{B})^l (A + x + \mathbf{a} \cdot \mathcal{B})^m \\
&= (-1)^m (x - A - \mathbf{a} \cdot \mathcal{B})^l (x - \mathbf{a} \cdot \mathcal{B})^m
\end{aligned}$$

using (2.8). The right-hand side of (5.5) is

$$(-1)^l (-x + \mathbf{a} \cdot \mathcal{B})^m (-x + A + \mathbf{a} \cdot \mathcal{B})^l = (-1)^m (x - \mathbf{a} \cdot \mathcal{B})^m (x - A - \mathbf{a} \cdot \mathcal{B})^l \tag{5.7}$$

and this proves the identity (5.5). The second requested identity (5.6) follows by differentiating (5.5). ☐

6. Some linear identities for the Bernoulli-Barnes numbers

This section contains proofs of some linear recurrences for the Bernoulli-Barnes numbers by the symbolic method discussed here. The first result appears as Theorem 5.5 in [1].

Theorem 6.1. *Let* $m \in \mathbb{N}$, $\mathbf{a} = (a_1, \cdots, a_n)$ *and* $A = a_1 + \cdots + a_n$. *Then*

$$B_{2m+1}(\mathbf{a}) = -\frac{1}{2(m+1)} \sum_{k=0}^{m} \binom{m+1}{k} (m+k+1) A^{m+1-k} B_{m+k}(\mathbf{a}) \tag{6.1}$$

and

$$\begin{aligned} B_{2m}(\mathbf{a}) \quad &= \quad -\frac{1}{(m+1)(2m+1)} \sum_{k=0}^{m-1} \binom{m+1}{k} (m+k+1) A^{m-k} B_{m+k}(\mathbf{a}) \\ &\quad + \frac{(2m)!}{A} \sum_{k=0}^{n-1} \sum_{|I|=k} \frac{B_{2m+1-n+k}(\mathbf{a}_I)}{(2m+1-n+k)!}. \end{aligned} \tag{6.2}$$

Proof. Start with the elementary identity

$$-(m+1)y^m(2y^{m+1} - (x+y)^m(x+2y)) = \sum_{k=0}^{m} \binom{m+1}{k} (m+k+1) x^{m+1-k} y^{m+k} \tag{6.3}$$

and denote the right-hand side by $f(y)$. Now use it with $x = A = a_1 + \cdots + a_n$ and $y = a_1\mathcal{B}_1 + \cdots + a_n\mathcal{B}_n = \mathbf{a} \cdot \mathcal{B}$ to obtain

$$\begin{aligned} f(\mathcal{B}) \quad &= \quad -(m+1)\left(2(\mathbf{a}\cdot\mathcal{B})^{2m+1} - (A + \mathbf{a}\cdot\mathcal{B})^m(A + 2\mathbf{a}\cdot\mathcal{B})(\mathbf{a}\cdot\mathcal{B})^m\right) \\ &= \quad -(m+1)\left(2(\mathbf{a}\cdot\mathcal{B})^{2m+1} - (A + \mathbf{a}\cdot\mathcal{B})^{m+1}(\mathbf{a}\cdot\mathcal{B})^m - (A + \mathbf{a}\cdot\mathcal{B})^m(\mathbf{a}\cdot\mathcal{B})^{m+1}\right). \end{aligned}$$

Then $\mathcal{B} = -\mathcal{B} - 1$ gives

$$(A+\mathbf{a}\cdot\mathcal{B})^{m+1}(\mathbf{a}\cdot\mathcal{B})^m = (-\mathbf{a}\cdot\mathcal{B})^{m+1}(-A-\mathbf{a}\cdot\mathcal{B})^m = -(\mathbf{a}\mathcal{B})^{m+1}(A+\mathbf{a}\cdot\mathcal{B})^m \tag{6.4}$$

that can be written as

$$(A + \mathbf{a}\cdot\mathcal{B})^{m+1}(\mathbf{a}\cdot\mathcal{B})^m + (\mathbf{a}\cdot\mathcal{B})^{m+1}(A + \mathbf{a}\cdot\mathcal{B})^m = 0. \tag{6.5}$$

The proof follows from here.

The second formula contains a small typo in the formulation given in [1]. To prove the corrected formula, use (2.2) with $x = 0$ and m replaced by $2m+1$ to obtain

$$-2B_{2m+1}(\mathbf{a}) = (2m+1)! \sum_{k=0}^{n-1} \sum_{|K|=k} \frac{B_{2m+1-n+k}(\mathbf{a}_K)}{(2m+1-n+k)!}. \tag{6.6}$$

The expression (6.1) for $B_{2m+1}(\mathbf{a})$ just established now gives

$$\begin{aligned} &\frac{(2m)!}{A} \sum_{k=0}^{n-1} \sum_{|K|=k} \frac{B_{2m+1-n+k}(\mathbf{a}_K)}{(2m+1-n+k)!} = \\ &\qquad \frac{1}{(m+1)(2m+1)} \sum_{k=0}^{m} \binom{m+1}{k} (m+1+k) A^{m-k} B_{m+k}(\mathbf{a}). \end{aligned}$$

Conclude with the observation that the term corresponding to $k = m$ in the last sum is $B_{2m}(\mathbf{a})$. Solving for it gives the stated expression. $\square$

The identity presented next appears as Theorem 1.1 in [1].

Theorem 6.2. *For* $n \geq 3$, $m \geq 1$ *odd and* $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$,
(6.7)

$$\sum_{j=n-m}^{n} \binom{n+j-4}{j-2} \frac{1}{(m-n+j)!} \sum_{|J|=j} B_{m-n+j}(\mathbf{a}_J) = \begin{cases} \frac{1}{2} & \textit{if } n = m = 3, \\ 0 & \textit{otherwise}, \end{cases}$$

where the inner sum is over all subsets $J \subset \{1, \cdots, n\}$ *of cardinality* j.

The proof presented next shows that Theorem 6.2 is part of a general class of identities. The proof also explains the appearance of the puzzling $\binom{n+j-4}{j-2}$.

Theorem 6.3. *Let* $\{\alpha_j^{(n)} : 1 \leq j \leq n\}$ *be a sequence of numbers satisfying the palindromic condition* $\alpha_{n-j}^{(n)} = \alpha_j^{(n)}$ *and let* f *be an odd function. Then*

$$\sum_{j=0}^{n} \alpha_j^{(n)} \sum_{|J|=j} f\left((\mathbf{a} \cdot \mathcal{B})_J - (\mathbf{a} \cdot \mathcal{B})_{J^*}\right) = 0, \tag{6.8}$$

where J^* *is the complement of* J *in* $\{1, \cdots, n\}$.

Proof. Observe that

$$(\mathbf{a} \cdot \mathcal{B})_J - (\mathbf{a} \cdot \mathcal{B})_{J^*} = -\left((\mathbf{a} \cdot \mathcal{B})_{J^*} - (\mathbf{a} \cdot \mathcal{B})_J\right) \tag{6.9}$$

and so for each term

$$\alpha_j^{(n)} f\left((\mathbf{a} \cdot \mathcal{B})_J - (\mathbf{a} \cdot \mathcal{B})_{J^*}\right) \tag{6.10}$$

in the sum (6.8), there is a corresponding term

$$\alpha_{n-j}^{(n)} f\left((\mathbf{a} \cdot \mathcal{B})_{J^*} - (\mathbf{a} \cdot \mathcal{B})_J\right) = \alpha_j^{(n)} f\left((\mathbf{a} \cdot \mathcal{B})_{J^*} - (\mathbf{a} \cdot \mathcal{B})_J\right). \tag{6.11}$$

The fact that f is an odd function implies

$$\alpha_j^{(n)} f\left((\mathbf{a} \cdot \mathcal{B})_{J^*} - (\mathbf{a} \cdot \mathcal{B})_J\right) + \alpha_{n-j}^{(n)} f\left((\mathbf{a} \cdot \mathcal{B})_{J^*} - (\mathbf{a} \cdot \mathcal{B})_J\right). \tag{6.12}$$

Hence the total sum over j vanishes. $\square$

Example 6.4. Theorem 6.2 corresponds to the choice

$$\alpha_j^{(n)} = \begin{cases} \binom{n-4}{j-2} & \text{if } 2 \leq j \leq n-2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.13}$$

To obtain this result start with the expansion

$$f((\mathbf{a} \cdot \mathcal{B})_K - (\mathbf{a} \cdot \mathcal{B})_{K^*}) = \sum_{j=0}^{|K^*|} \sum_{|J|=j} |a_J| f^{(j)}((\mathbf{a} \cdot \mathcal{B})_{J^*}) \tag{6.14}$$

and then

$$\begin{aligned} \sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)} f\left((\mathbf{a} \cdot \mathcal{B})_K - (\mathbf{a} \cdot \mathcal{B})_{K^*}\right) &= \sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)} \sum_{j=0}^{|K^*|} \sum_{|J|=j} |\mathbf{a}|_J f^{(j)}((\mathbf{a} \cdot \mathcal{B})_{J^*}) \\ &= \sum_{j=0} \sum_{|J|=j} |\mathbf{a}_J| f^{(j)}\left((\mathbf{a} \cdot \mathcal{B})_{J^*}\right) \sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)}. \end{aligned}$$

Now

$$\sum_{|K|=k} 1 = \binom{n-j}{k} \tag{6.15}$$

since there are $\binom{n-j}{k}$ subsets of K of size k in $\{1, \cdots, n\}$ that do not overlap with J. Hence

$$\sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_j^{(n)} = \sum_{k=2}^{n-2} \binom{n-4}{k-2} \binom{n-j}{n-j-k} = \binom{2n-j-4}{n-j-2} \tag{6.16}$$

by the Chu-Vandermonde identity [4, p. 169]. This gives

$$\sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)} f\left((\mathbf{a} \cdot \mathcal{B})_K - (\mathbf{a} \cdot \mathcal{B})_{K^*}\right) = \sum_{j=0}^{n} \binom{2n-j-4}{n-j-4} \sum_{|J|=j} |\mathbf{a}|_J f^{(j)}\left((\mathbf{a} \cdot \mathcal{B})_{J^*}\right).$$

The change of summation variable $j \mapsto n-j$ has the effect

$$\binom{2n-j-4}{n-j-2} \mapsto \binom{n+j-4}{j-2} \tag{6.17}$$

and this produces Theorem 6.2 by taking $f(x) = x^m/m!$.

7. One final recurrence for the Bernoulli-Barnes numbers

Identities between generalized Bernoulli-Barnes numbers of different orders are rare in the literature. The symbolic method used in this paper provides an efficient way to prove and generalize such identities, as shown in the cases studied in the previous sections. However, other techniques may compete favorably. This last section provides a new occurrence of these identities and purely analytical proofs are provided.

The exponential generating function for the Bernoulli-Barnes polynomials in the special case of parameter $\mathbf{1} = (1, \cdots, 1) \in \mathbb{C}^n$, is given in (1.5) by

$$\sum_{j=0}^{\infty} B_j^{(n)}(x; \mathbf{1}) \frac{z^j}{j!} = e^{xz} \frac{z^n}{(e^z-1)^n}, \tag{7.1}$$

where the parameter n counts the length of $\mathbf{1} \in \mathbb{C}^n$. Introduce the notation

$$B_j^{(n)}(x) = B_j^{(n)}(x; \mathbf{1}), \tag{7.2}$$

and write (7.1) as

$$\sum_{j=0}^{\infty} B_j^{(n)}(x) \frac{z^j}{j!} = e^{xz} \frac{z^n}{(e^z-1)^n}, \tag{7.3}$$

This special case of Bernoulli-Barnes polynomials is also known as Nörlund polynomials.

A connection between hypergeometric function and these polynomials is now made explicit. The identity

$${}_2F_1\left(\left.\begin{matrix} 1 \quad 1 \\ p+2 \end{matrix}\right| z\right) = \frac{p+1}{z} \left[\sum_{\ell=0}^{p-1} \frac{1}{(p-\ell)} \left(\frac{z-1}{z}\right)^{\ell} - \left(\frac{z-1}{z}\right)^{p} \log(1-z)\right] \tag{7.4}$$

for the hypergeometric function

$$ {}_2F_1\left(\begin{matrix} 1 & 1 \\ p+2 & \end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(p+2)_n}\frac{z^n}{n!} \tag{7.5}$$

can be found in [5, 7.3.1.136]. The substitution $z \mapsto 1-e^z$ gives

$$ {}_2F_1\left(\begin{matrix} 1 & 1 \\ p+2 & \end{matrix}\middle| 1-e^z\right) = (p+1)\left[\frac{ze^{pz}}{(e^z-1)^{p+1}} - \sum_{\ell=0}^{p-1}\frac{1}{(p-\ell)}\frac{e^{\ell z}}{(e^z-1)^{\ell+1}}\right]. \tag{7.6}$$

The terms in the sum above are now written in terms of the Bernoulli-Barnes polynomial. To start, (7.1) gives

$$\frac{ze^{pz}}{(e^z-1)^{p+1}} = z^{-p}e^{pz}\left(\frac{z}{e^z-1}\right)^{p+1} = z^{-p}\sum_{j=0}^{\infty} B_j^{(p+1)}(p)\frac{z^j}{j!} = \sum_{j=-p}^{\infty}\frac{B_{j+p}^{(p+1)}(p)}{(j+p)!}z^j$$

for the first term in (7.6). The second term in (7.6) can be written as

$$\sum_{\ell=0}^{p-1}\frac{1}{(p-\ell)}\frac{e^{\ell z}}{(e^z-1)^{\ell+1}} = \sum_{\ell=0}^{p-1}\frac{1}{(p-\ell)}\sum_{j=-\ell-1}^{\infty} B_{j+\ell+1}^{(\ell+1)}(\ell)\frac{z^j}{(j+\ell+1)!}. \tag{7.7}$$

Since the hypergeometric function is analytic at $z=0$, the coefficients of negative powers on the right-hand side of (7.6) must vanish. This leads, for $-p \leq j \leq -1$, to the identity

$$\frac{B_{j+p}^{(p+1)}(p)}{(j+p)!} = \sum_{\ell=-j-1}^{p-1}\frac{1}{p-\ell}\frac{B_{j+\ell+1}^{(\ell+1)}(\ell)}{(j+\ell+1)!}. \tag{7.8}$$

A shift in the index and denoting $j+p$ by r produces the final statement.

Theorem 7.1. *Let* $0 \leq r \leq p-1$. *Then*

$$\frac{B_r^{(p+1)}(p)}{r!} = \sum_{k=1}^{r+1}\frac{1}{k}\frac{B_{r+1-k}^{(p+1-k)}(p-k)}{(r+1-k)!}, \tag{7.9}$$

or

$$\frac{B_r^{(p+1)}(p)}{r!} - \frac{B_r^{(p)}(p-1)}{r!} = \sum_{k=1}^{r}\frac{1}{(k+1)}\frac{B_{r-k}^{(p-k)}(p+1-k)}{(r-k)!}. \tag{7.10}$$

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