THE 2-ADIC VALUATION OF A SEQUENCE ARISING FROM A RATIONAL INTEGRAL

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Abstract. We analyze properties of the 2-adic valuation of an integer sequence that originates from an explicit evaluation of a quartic integral. We also give a combinatorial interpretation of the valuations of this sequence.

1. Introduction

Wallis's formula

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m} \tag{1.1}$$

is one of the earlier instances of evaluation of definite integrals where the result contains interesting arithmetical and combinatorial properties. In this paper we examine such connection for the integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{1.2}$$

The condition $a > -1$ is imposed for convergence. The evaluation

$$N_{0,4}(a,m) = \frac{\pi}{2}\frac{P_m(a)}{[2(a+1)]^{m+\frac{1}{2}}} \tag{1.3}$$

where

$$P_m(a) = \sum_{l=0}^{m} d_l(m)a^l \tag{1.4}$$

with

$$d_l(m) = 2^{-2m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}, \quad 0 \leq l \leq m, \tag{1.5}$$

appeared in [4]. The reader will find in [2] a survey of the different proofs of (1.3) and an introduction to the many issues involved in the evaluation of definite integrals in [8].

The study of combinatorial aspects of the sequence $d_l(m)$ was initiated in [3] where the authors show that $d_l(m)$ form a *unimodal* sequence, that is, there exists and index l^* such that $d_0(m) \leq \ldots \leq d_{l^*}(m)$ and $d_{l^*}(m) \geq \ldots \geq d_m(m)$. The fact that $d_l(m)$ satisfies the stronger condition of *logconcavity* $d_{l-1}(m)d_{l+1}(m) \leq d_l^2(m)$

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has been recently established in [6]. We consider here arithmetical properties of the sequence $d_{l,m}$. It is more convenient to analyze the auxiliary sequence

$$A_{l,m} = l!\,m!\,2^{m+l}d_{l,m} = \frac{l!\,m!}{2^{m-l}}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l} \tag{1.6}$$

for $m \in \mathbb{N}$ and $0 \leq l \leq m$. The integral (1.2) is then given explicitly as

$$N_{0,4}(a;m) = \frac{\pi}{\sqrt{2}\,m!\,(4(2a+1))^{m+1/2}}\sum_{l=0}^{m} A_{l,m}\frac{a^l}{l!}. \tag{1.7}$$

In [5] it is shown that $A_{l,m} \in \mathbb{N}$. Observe that the computation of $A_{l,m}$ using (1.6) is more efficient if l is close to m. For instance,

$$A_{m,m} = 2^m(2m)! \text{ and } A_{m-1,m} = 2^{m-1}(2m-1)!(2m+1). \tag{1.8}$$

A second method to compute $A_{l,m}$, efficient now when l is small, has been discussed in [5]. There, it is shown that $A_{l,m}$ is a linear combination (with polynomial coefficients) of

$$\prod_{k=1}^{m}(4k-1) \text{ and } \prod_{k=1}^{m}(4k+1). \tag{1.9}$$

For example,

$$A_{0,m} = \prod_{k=1}^{m}(4k-1) \text{ and } A_{1,m} = (2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1). \tag{1.10}$$

The results described in this paper started with some empirical observations on the behavior of the 2-adic valuation of $A_{l,m}$, i.e. $\nu_2(A_{l,m})$. Recall that, for $x \in \mathbb{N}$, the 2-adic valuation $\nu_2(x)$ is the highest power of 2 that divides x. This is extended to $x = a/b \in \mathbb{Q}$ via $\nu_2(x) = \nu_2(a) - \nu_2(b)$. From (1.10) it follows that $A_{0,m}$ is odd, so $\nu_2(A_{0,m}) = 0$. Moreover,

$$\nu_2(A_{1,m}) = \nu_2(m(m+1)) + 1, \tag{1.11}$$

i.e., the main result of [5]. We present as Theorem 2.1, an expression for $\nu_2(A_{l,m})$ that generalizes (1.11).

The study of the sequence

$$X(l) := \{\nu_2(A_{l,l+m-1}) : m \geq 1\} \tag{1.12}$$

requires the introduction of two operators, F and T, defined in (4.1) and (4.2), respectively. The iteration of these operators creates an integer vector

$$\Omega(l) \quad := \quad \left\{n_1,\, n_2,\, n_3, \cdots, n_{\omega(l)}\right\}, \quad \text{with } n_i \in \mathbb{N}, \tag{1.13}$$

associated to the index $l \in \mathbb{N}$. We call $\Omega(l)$ the *reduction sequence* of l. See (4.2) for the precise definition of the integers n_j. The structure of $X(l)$ motivates the following definition.

Definition 1.1. Let $s \in \mathbb{N}$, $s \geq 2$. We say that a sequence $\{a_j : j \in \mathbb{N}\}$ is *simple of length* s (or s-*simple*) if s is the largest integer such that for each $t \in \{0, 1, 2, \cdots\}$, we have

$$a_{st+1} = a_{st+2} = \cdots = a_{s(t+1)}. \tag{1.14}$$

The sequence $\{a_j : j \in \mathbb{N}\}$ is said to have a *block structure* if it is s-simple for some $s \geq 2$.

Section 2 presents two proofs of the expression for $\nu_2(A_{l,m})$. Section 3 shows that $X(l)$ is a simple sequence of length $2^{1+\nu_2(l)}$. In Section 4 an algorithm generating the vector $\Omega(l)$ is described in detail. A combinatorial interpretation of $\Omega(l)$, as the composition of l, is provided in Section 5. Theorem 5.5 gives $\Omega(l)$ in terms of the dyadic expansion of l. More precisely, if $\{k_1, \cdots, k_n : 0 \leq k_1 < k_2 < \cdots < k_n\}$ is the unique collection of distinct nonnegative integers such that $l = \sum_{i=1}^{n} 2^{k_i}$, then the reduction sequence $\Omega(l)$ of l is $\{k_1+1, k_2-k_1, \cdots, k_n-k_{n-1}\}$. Finally, the last section contains a conjecture on symmetries of the graph of $\nu_2(A_{l,m})$.

2. The 2-adic valuation of $A_{l,m}$

In this section we prove that $\nu_2(A_{l,m})$ agrees with $\nu_2((m+1-l)_{2l})+l$. The first proof actually produces the latter term in a natural way starting from the former. The second proof employs the WZ-machinery [9] to prove the identity (2.1).

Theorem 2.1. *The* 2*-adic valuation of* $A_{l,m}$ *satisfies*

$$\nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l, \tag{2.1}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ *is the Pochhammer symbol for* $k \geq 1$. *For* $k=0$*, we define* $(a)_0 = 1$.

Proof. **First proof**. We have

$$\nu_2\left(A_{l,m}\right) = l + \nu_2\left(\sum_{k=l}^{m} T_{m,k}\,\frac{(m+k)!}{(m-k)!\,(k-l)!}\right), \tag{2.2}$$

where

$$T_{m,k} = \frac{(2m-2k)!}{2^{m-k}\,(m-k)!}. \tag{2.3}$$

The identity

$$T_{m,k} = \frac{(2(m-k))!}{2^{m-k}\,(m-k)!} = (2m-2k-1)(2m-2k-3)\cdots 3\cdot 1 \tag{2.4}$$

shows that $T_{m,k}$ is an odd integer. Then (2.2) can be written as

$$\begin{aligned} \nu_2(A_{l,m}) &= l + \nu_2\left(\sum_{k=0}^{m-l} T_{m,l+k}\,\frac{(m+k+l)!}{(m-k-l)!\,k!}\right) \\ &= l + \nu_2\left(\sum_{k=0}^{m-l} T_{m,l+k}\frac{(m-k-l+1)_{2k+2l}}{k!}\right). \end{aligned}$$

The term corresponding to $k=0$ is singled out as we write

$$\nu_2(A_{l,m}) = l + \nu_2\left(T_{m,l}(m-l+1)_{2l} + \sum_{k=1}^{m-l} T_{m,l+k}\frac{(m-k-l+1)_{2k+2l}}{k!}\right).$$

The claim

$$\nu_2\left(\frac{(m-k-l+1)_{2k+2l}}{k!}\right) > \nu_2((m-l+1)_{2l}) \tag{2.5}$$

for any k, $1 \leq k \leq m-l$, will complete the proof.

To prove (2.5) we use the identity

$$\frac{(m-k-l+1)_{2k+2l}}{k!} = (m-l+1)_{2l} \cdot \frac{(m-l-k+1)_k\,(m+l+1)_k}{k!}$$

and the fact that the product of k consecutive numbers is always divisible by $k!$. This follows from the identity

$$\frac{(a)_k}{k!} = \binom{a+k-1}{k}. \tag{2.6}$$

Now if $m+l$ is odd,

$$\nu_2\left(\frac{(m-l-k+1)_k}{k!}\right) \geq 0 \text{ and } \nu_2((m+l+1)_k) > 0, \tag{2.7}$$

and if $m+l$ is even

$$\nu_2\left(\frac{(m+l+1)_k}{k!}\right) \geq 0 \text{ and } \nu_2((m-l-k+1)_k) > 0. \tag{2.8}$$

This proves (2.5) and establishes the theorem.

Second proof. Define the numbers

$$B_{l,m} := \frac{A_{l,m}}{2^l(m+1-l)_{2l}}. \tag{2.9}$$

We need to prove that $B_{l,m}$ is odd. The WZ-method [9] provides the recurrence

$$B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m-1.$$

Since the initial values $B_{m,m} = 1$ and $B_{m-1,m} = 2m+1$ are odd, it follows that $B_{l,m}$ is an odd integer. □

3. Properties of the function $\nu_2(A_{l,m})$

Let $l \in \mathbb{N} \cup \{0\}$ be fixed. In this section we describe properties of the function $\nu_2(A_{l,m})$. In particular, we show that each of these sequences has a block structure.

Theorem 3.1. *Let $l \in \mathbb{N} \cup \{0\}$ be fixed. Then for $m \geq l$, we have*

$$\nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2(m+l+1) - \nu_2(m-l+1). \tag{3.1}$$

Proof. From (2.1) and $(a)_k = (a+k-1)!/(a-1)!$, we have

$$\nu_2(A_{l,m}) = \nu_2\left(\frac{(m+l)!}{(m-l)!}\right) + l. \tag{3.2}$$

This implies

$$\begin{aligned}
\nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) &= \nu_2\left(\frac{(m+l+1)!}{(m-l+1)!}\right) - \nu_2\left(\frac{(m+l)!}{(m-l)!}\right) \\
&= \nu_2\left(\frac{(m+l+1)!\,(m-l)!}{(m-l+1)!\,(m+l)!}\right) \\
&= \nu_2(m+l+1) - \nu_2(m-l+1).
\end{aligned}$$

□

The next corollary is a special case of Theorem 3.1.

Corollary 3.2. *The sequence* $\nu_2(A_{l,m})$ *satisfies*
1) $\nu_2(A_{l,l+1}) = \nu_2(A_{l,l})$.
2) *For* l *even,*

$$\nu_2(A_{l,l+3}) = \nu_2(A_{l,l+2}) = \nu_2(A_{l,l+1}) = \nu_2(A_{l,l}).$$

3) *The sequence* $\nu_2(A_{1,m})$ *is 2-simple; i.e.,* $\nu_2(A_{1,m+1}) = \nu_2(A_{1,m})$ *for* m *odd. In fact,*

$$A_{1,m} = \{2,\, 2,\, 3,\, 3,\, 2,\, 2,\, 4,\, 4,\, 2,\, 2, \ldots\}.$$

Fix $k,\, l \in \mathbb{N}$ and let $\mu := 1 + \nu_2(l)$. Define the following sets

$$C_{k,l} := \{ l + k \cdot 2^{\mu} + j \,:\, 0 \leq j \leq 2^{\mu} - 1 \}, \tag{3.3}$$

which will be instrumental in proving the main result of this section; i.e., $\{\nu_2(A_{l,m})\}$ is $2^{1+\nu_2(l)}$-simple.

We begin by showing that these sets form a partition of $\mathbb{N}$. Moreover, for fixed $k,\, l \in \mathbb{N}$ the set $C_{k,l}$ has cardinality 2^{μ} and the 2-adic valuation of $\{A_{l,m} \,:\, m \in C_{k,l}\}$ is constant. For example, if $l \in \mathbb{N}$ is odd, then $\mu = 1$ and

$$C_{k,l} = \{l + 2k,\, l + 2k + 1\}. \tag{3.4}$$

The next result is immediate.

Lemma 3.3. *Let* $l \in \mathbb{N}$ *be fixed. The sets* $\{C_{k,l} \,:\, k \geq 0\}$ *form a disjoint partition of* $\mathbb{N}$*; namely,*

$$\{m \in \mathbb{N} \,:\, m \geq l\} = \bigcup_{k \geq 0} C_{k,l}, \tag{3.5}$$

and $C_{r,l} \cap C_{t,l} = \emptyset$*, whenever* $r \neq t$.

Lemma 3.4. *Fix* $l \in \mathbb{N}$ *and let* $\mu = \nu_2(2l)$.
1) *The sequence* $\{\nu_2(A_{l,m}) \,:\, m \in C_{k,l}\}$ *is constant. We denote this value by* $\nu_2(C_{k,l})$.
2) *For* $k \geq 0$, $\nu_2(C_{k+1,l}) \neq \nu_2(C_{k,l})$.

Proof. Suppose $0 \leq j \leq 2^{\mu} - 2$. Since $\nu_2(2l) = \mu \leq \nu_2(k \cdot 2^{\mu})$, then

$$\nu_2(2l + k \cdot 2^{\mu}) \geq \nu_2(2l) = \mu > \nu_2(j+1), \tag{3.6}$$

because $j + 1 < 2^{\mu}$. Therefore

$$\nu_2(2l + k \cdot 2^{\mu} + j + 1) = \nu_2(j+1) = \nu_2(k \cdot 2^{\mu} + j + 1). \tag{3.7}$$

Using these facts and (3.1), we obtain

$$\begin{aligned} \nu_2(A_{l,l+k\cdot 2^{\mu}+j+1}) - \nu_2(A_{l,l+k\cdot 2^{\mu}+j}) &= \nu_2(2l + k \cdot 2^{\mu} + j + 1) - \nu_2(k \cdot 2^{\mu} + j + 1) \\ &= \nu_2(j+1) - \nu_2(j+1) = 0 \end{aligned}$$

for consecutive values in $C_{k,l}$. This proves part 1). To prove part 2), it suffices to take elements $l + k \cdot 2^{\mu} + 2^{\mu} - 1 \in C_{k,l}$ and $l + (k+1) \cdot 2^{\mu} \in C_{k+1,l}$ and compare their 2-adic values. Again by (3.1), we have

$$\begin{aligned} \nu_2(A_{l,l+(k+1)\cdot 2^{\mu}}) - \nu_2(A_{l,l+(k+1)\cdot 2^{\mu}-1}) &= \nu_2(2l + (k+1) \cdot 2^{\mu}) - \nu_2((k+1) \cdot 2^{\mu}) \\ &= \mu + \nu_2(2l \cdot 2^{-\mu} + k + 1) - \mu - \nu_2(k+1) \\ &= \nu_2(2l \cdot 2^{-\mu} + k + 1) - \nu_2(k+1) \neq 0. \end{aligned}$$

The last step follows from $2l \cdot 2^{-\mu}$ being odd and thus $2l \cdot 2^{-\mu} + k + 1$ and $k + 1$ having opposite parities. This completes the proof. $\square$

Theorem 3.5. *For each $l \geq 1$, the set $\{\nu_2(A_{l,m}) : m \geq l\}$ is an s-simple sequence, with $s = 2^{1+\nu_2(l)}$.*

Proof. From Lemma 3.3 and Lemma 3.4, we know that $\nu_2(\cdot)$ maintains a constant value on each of the disjoint sets $C_{k,l}$. The length of each of these blocks is $2^{1+\nu_2(l)}$. □

4. The algorithm and its combinatorial interpretation

In this section we describe an algorithm that extracts from the sequence $X(1) := \{\nu_2(A_{1,m}) : m \geq 1\}$ its combinatorial information. We begin with the definition of the operators F and T mentioned in the Introduction.

Definition 4.1. The maps F and T. These are defined by

$$F(\{a_1, a_2, a_3, \cdots\}) \quad := \quad \{a_1, a_1, a_2, a_3, \cdots\}, \tag{4.1}$$

and

$$T(\{a_1, a_2, a_3, \cdots\}) \quad := \quad \{a_1, a_3, a_5, a_7, \cdots\}. \tag{4.2}$$

We employ the notation

$$c \quad := \quad \{\nu_2(m) : \quad m \geq 1\} = \{0, 1, 0, 2, 0, 1, 0, 3, 0, \cdots\}. \tag{4.3}$$

The algorithm:

1) Start with the sequence $X(l) := \{\nu_2(A_{l,l+m-1}) : \quad m \geq 1\}$.

2) Find $n \in \mathbb{N}$ so that the sequence $X(l)$ is 2^n-simple. Define $Y(l) := T^n(X(l))$. At the initial stage, Theorem 3.5 ensures that $n = 1 + \nu_2(l)$.

3) Introduce the shift $Z(l) := Y(l) - c$.

4) Define $W(l) := F(Z(l))$.

If $W(l)$ is a constant sequence, then STOP; otherwise go to step 2) with W instead of X. Define $X_k(l)$ as the new sequence at the end of the $(k-1)$th cycle of this process, with $X_1(l) = X(l)$.

Section 5 contains the justification for the steps of this algorithm. In particular, we prove that the sequences $X_k(l)$ have a block structure, so they can be used back in step 1 after each cycle. Theorem 5.3 states that the algorithm finishes in a finite number of steps and that $W(l)$ is essentially $X(j)$, for some $j < l$.

Definition 4.2. Let $\omega(l)$ be the number of cycles required for the algorithm to yield a constant sequence and denote by n_j the integers appearing in Step 2 of the algorithm. The integer vector

$$\Omega(l) \quad := \quad \left\{n_1, n_2, n_3, \cdots, n_{\omega(l)}\right\} \tag{4.4}$$

is called the *reduction sequence* of l. The number $\omega(l)$ will be called the *reduction length* of l. The constant sequence obtained after $\omega(l)$ cycles is called the *reduced constant*.

TABLE 1. Reduction sequence for $1 \leq l \leq 15$.

l	binary form	$\Omega(l)$
4	100	3
5	101	1, 2
6	110	2, 1
7	111	1, 1, 1
8	1000	4
9	1001	1, 3
10	1010	2, 2
11	1011	1, 1, 2
12	1100	3, 1
13	1101	1, 2, 1
14	1110	2, 1, 1
15	1111	1, 1, 1, 1

In Corollary 5.8 we enumerate $\omega(l)$ as the number of ones in the binary expansion of l. Therefore the algorithm yields a constant sequence in a finite number of steps. In fact, the algorithm terminates after $O(\log_2(l))$ cycles as will follow directly from Corollory 5.8. Table 1 shows the results of the algorithm for $4 \leq l \leq 15$.

We now provide a combinatorial interpretation of $\Omega(l)$. This requires the composition of the index l.

Definition 4.3. Let $l \in \mathbb{N}$. The *composition* of l, denoted by $\Omega_1(l)$, is defined as follows: write l in binary form. Read the sequence from right to left. The first part of $\Omega_1(l)$ is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on.

Example 4.4. Reading off the values from Table 1, we obtain $\Omega_1(13) = \{1, 2, 1\}$ and $\Omega_1(14) = \{2, 1, 1\}$. Therefore $\Omega_1(13) = \Omega(13)$ and $\Omega_1(14) = \Omega(14)$. Corollary 5.6 shows that this is always true.

The next result describes the formation of $\Omega_1(l)$ from $\Omega_1(\lfloor l/2 \rfloor)$.

Lemma 4.5. *Given the values of $\Omega_1(l)$ for $2^j \leq l \leq 2^{j+1} - 1$, the list for $2^{j+1} \leq l \leq 2^{j+2} - 1$ is formed according to the following rule:*

l is even: add 1 to the first part of $\Omega_1(l/2)$ to obtain $\Omega_1(l)$;

l is odd: prepend a 1 to $\Omega_1\left(\frac{l-1}{2}\right)$ to obtain $\Omega_1(l)$.

Proof. Let $x_1x_2\cdots x_t$ be the binary representation of l. Then $x_1x_2\cdots x_t0$ corresponds to $2l$. Thus, the first part of $\Omega_1(2l)$ is increased by 1, due to the extra 0 on the right. The relative position of the remaining 1s stays the same. A similar argument takes care of $\Omega_1(2l+1)$. The extra 1 that is placed at the end of the binary representation gives the first 1 in $\Omega_1(2l+1)$. □

We now relate the 2-adic valuation of $A_{l,m}$ to that of $A_{\lfloor l/2 \rfloor,m}$.

Proposition 4.6. *Let*

$$\lambda_l := \frac{1-(-1)^l}{2}, \quad M_0 := \lfloor \frac{m+\lambda_l}{2} \rfloor. \tag{4.5}$$

Then

$$\nu_2(A_{l,m}) = 2l - \lfloor l/2 \rfloor + \lambda_l \nu_2(M_0 - \lfloor l/2 \rfloor) + \nu_2(A_{\lfloor l/2 \rfloor, M_0}). \tag{4.6}$$

Proof. We present the details for $\nu_2(A_{2l,2m})$. Theorem 2.1 gives

$$\begin{aligned}
\nu_2(A_{2l,2m}) &= \nu_2((2m-2l+1)_{4l}) + 2l \\
&= \nu_2((2m-2l+1)(2m-2l+2)\cdots(2m+2l-1)(2m+2l)) + 2l \\
&= \nu_2(2^{2l}(m-l+1)(m-l+2)\cdots(m+l)) + 2l \\
&= 4l + \nu_2((m-l+1)_{2l}) \\
&= 3l + \nu_2(A_{l,m}).
\end{aligned}$$

A similar calculation shows that

$$\nu_2(A_{2l+1,2m}) = 3l + 2 + \nu_2(A_{l,m}) + \nu_2(m-l). \tag{4.7}$$

The general case then follows from Theorem 3.1. □

Corollary 4.7. *The 2-adic valuation of $A_{l,m}$ satisfies*

$$\nu_2(A_{l,m}) = 2l + \nu_2(l!) + \sum_{k \geq 0} \lambda_{\lfloor l/2^k \rfloor} \, \nu_2(M_k - \lfloor l/2^{k+1} \rfloor) \tag{4.8}$$

where

$$M_k = \lfloor \frac{m + \lambda_l + 2\lambda_{\lfloor l/2 \rfloor} + \cdots + 2^k \lambda_{\lfloor l/2^k \rfloor}}{2^{1+k}} \rfloor = \lfloor \frac{m + \sum_{n=0}^{k} 2^n \lambda_{\lfloor l/2^n \rfloor}}{2^{1+k}} \rfloor. \tag{4.9}$$

Proof. This is a repeated application of Proposition 4.6. The first term results from

$$\begin{aligned}
\sum_{k \geq 0} \left(2 \lfloor \frac{l}{2^k} \rfloor - \lfloor \frac{l}{2^{k+1}} \rfloor \right) &= 2l + \sum_{k \geq 1} \lfloor \frac{l}{2^k} \rfloor \\
&= 2l + \nu_2(l!).
\end{aligned}$$

□

5. Verification of the Algorithm and the Reduction sequence

In this section we show that the algorithm presented in Section 4 terminates after a finite numbers of cycles. Moreover, we prove that $\Omega(l)$, the reduction sequence of l, is identical to the composition sequence of l.

Notation: The constant sequences will be denoted by $(t) = \{t, t, t, \dots\}$.

Definition 5.1. A sequence $(a) = \{a_1, a_2, a_3, \dots\}$ is a *translate* of $(b) = \{b_1, b_2, b_3, \dots\}$ if $(a) = (b) + (t)$, for some constant sequence (t). Addition of sequences is performed term by term.

We first consider the base case $l = 1$.

Lemma 5.2. *The initial case $l = 1$ satisfies*

$$W(1) = F(T(X(1)) - c) = (2), \tag{5.1}$$

where (c) is given in (4.3).

Proof. Since $\nu_2(A_{1,m}) = \nu_2(m(m+1)) + 1$ and $\nu_2(2m-1) = 0$, we have

$$T(X(1)) = \{\nu_2((2m-1)(2m)) + 1 : m \geq 1\} = \{\nu_2(m) + 2 : m \geq 1\} = c + (2).$$

Then the assertion follows from $F((t)) = (t)$ for a constant (t). □

Theorem 5.3. *The algorithm terminates after finitely many iterations. Furthermore, in each cycle, $W(l)$ is a translate of $X(j)$, for some $j < l$.*

Proof. Start by rewriting the terms in $X(l)$ as

$$\nu_2\left(\frac{(m-1+2l)!}{(m-1)!}\right) + l = \nu_2((m-1+2l)(m-2+2l)\cdots(m+1)m) + l, \qquad m \geq 1.$$

Then, the operator T acts on these to yield (for $m \geq 1$)

$$\begin{aligned} &\nu_2((2m-2+2l)(2m-3+2l)\cdots(2m)(2m-1)) + l \\ &= \nu_2((m-1+l)\cdots(m)) + 2l \\ &= \nu_2\left(\frac{(m-1+l)!}{(m-1)!}\right) + 2l. \end{aligned} \tag{5.2}$$

Case I: l is even. From (5.2), we can easily obtain the relation

$$T(X(l)) = \{\nu_2\left(\frac{(m-1+l)!}{(m-1)!}\right) + l/2 + t : m \geq 1\} = X(l/2) + (t), \qquad t = 3l/2.$$

Case II: l is odd. Upon subtracting the sequence $c = \{\nu_2(m) : m \geq 1\}$ from (5.2) we get that

$$\nu_2\left(\frac{(m+l-1)!}{m!}\right) + 2l = \nu_2\left(\frac{(m+l-1)!}{m!}\right) + \frac{l-1}{2} + \frac{3(l-1)}{2} + 2,$$

for $m \geq 1$. Then, apply the operator F to the last sequence and find

$$W(l) = \{\nu_2\left(\frac{(m-2+l)!}{(m-1)!}\right) + \frac{l-1}{2} + t : m \geq 1\} = X\left(\frac{l-1}{2}\right) + (t), \qquad t = (3l+1)/2.$$

Here, we have utilized the property that $\nu_2(r!) = \nu_2((r-1)!)$, when $r \geq 1$ is odd. This justifies that the first term augmented in the sequence, as a result of the action of F, coincides with the next term (these are values at $m = 1$ and $m = 2$, respectively).

We can now conclude that in either of the two cases (or a combination thereof), the index l shrinks dyadically. Thus the reduction algorithm must end in a finite step into a translate of $X(1)$. Since Lemma 5.2 handles $X(1)$, the proof is completed. □

Corollary 5.4. *For general $k \in \mathbb{N}$, the sequence $X_k(l)$ is 2^{n_k}-simple for some $n_k \in \mathbb{N}$.*

Theorem 5.5. *Let* $\{k_1, \cdots, k_n : 0 \leq k_1 < k_2 < \cdots < k_n\}$, *be the unique collection of distinct nonnegative integers such that*

$$l = \sum_{i=1}^{n} 2^{k_i}. \tag{5.3}$$

Then the reduction sequence $\Omega(l)$ *of* l *is* $\{k_1 + 1, \, k_2 - k_1, \cdots, k_n - k_{n-1}\}$.

Proof. The argument of the proof is to check that the rules of formation for $\Omega_1(l)$ also hold for the reduction sequence $\Omega(l)$. The proof is divided according to the parity of l. The case l odd starts with $l = 1$, where the block length is 2. From Theorem 2.1 we obtain a constant sequence after iterating the algorithm once. Thus the algorithm terminates and the reduction sequence for $l = 1$ is $\Omega(1) = \{1\}$.

Now consider the general even case: $X(2l)$. Theorem 5.3 shows that applying T to this sequence yields a translate of $X(l)$. This does not affect the reduction sequence $\Omega(l)$, but the doubling of block length increases the first term of $\Omega(l)$ by 1. Therefore

$$\Omega(2l) = \{k_1 + 2, \, k_2 - k_1, \, \cdots, k_n - k_{n-1}\}. \tag{5.4}$$

This is precisely what happens to the binary digits of l: if

$$l = \sum_{i=1}^{n} 2^{k_i}, \text{ then } 2l = \sum_{i=1}^{n} 2^{k_i+1}.$$

This concludes the argument for even indices.

For the general odd case, $X(2l+1)$, we apply T, subtract c and then apply F. Again, by Theorem 5.3, this gives us a translate of $X(l)$. We conclude that, if the reduction sequence of l is

$$\{k_1 + 1, \, k_2 - k_1, \cdots, k_n - k_{n-1}\}, \tag{5.5}$$

then that of $2l + 1$ is

$$\{1, \, k_1 + 1, \, k_2 - k_1, \cdots, k_n - k_{n-1}\}. \tag{5.6}$$

This is precisely the behavior of Ω_1. The proof is complete. □

Corollary 5.6. *The reduction sequence* $\Omega(l)$ *associated to an integer* l *is the sequence of compositions of* l, *that is,*

$$\Omega(l) = \Omega_1(l). \tag{5.7}$$

Corollary 5.7. *The reduced constant is* $2l + \nu_2(l!) = \nu_2(A_{l,l})$.

Proof. In Corollary 4.7, subtract the last term as per the reduction algorithm. □

Corollary 5.8. *The set* $\Omega(l)$ *has cardinality*

$$s_2(l) = \text{the number of ones in the binary expansion of } l. \tag{5.8}$$

Note. The function $s_2(l)$ defined in (5.8) has recently appeared in a different divisibility problem. Lengyel [7] conjectured, and De Wannemacker [10] proved, that the 2-adic valuation of the Stirling numbers of the second kind $S(n,k)$ is given by

$$\nu_2(S(2^n, k)) = s_2(k) - 1. \tag{5.9}$$

The reader will find in [1] a general study of the 2-adic valuation of Stirling numbers.

6. A symmetry conjecture on the graphs of $\nu_2(A_{l,m})$

The graphs of the function $\nu_2(A_{l,m})$, where we take every other $2^{1+\nu_2(l)}$-element to reduce the repeating blocks to a single value, are shown in the next figures. We conjecture that these graphs have a symmetry property generated by what we call an *initial segment* from which the rest is determined by adding a *central piece* followed by a *folding rule*. We conclude with sample pictures of this phenomenon.

Example 6.1. For $l = 1$, the first few values of the reduced table are

$$\{2,\ 3,\ 2,\ 4,\ 2,\ 3,\ 2,\ 5,\ 2,\ 3, \ldots\}.$$

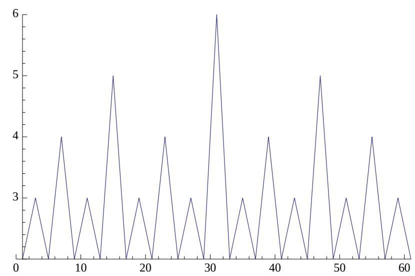

Figure 1. The 2-adic valuation of $A_{1,m}$

The ingredients are:

initial segment: $\{2,\ 3,\ 2\}$,

central piece: the value at the center of the initial segment, namely 3.

rules of formation: start with the initial segment and add 1 to the central piece and reflect.

This produces the sequence

$$\{2,3,2\} \to \{2,3,2,4\} \to \{2,3,2,4,2,3,2\} \to \{2,3,2,4,2,3,2,5\} \to$$
$$\to \{2,3,2,4,2,3,2,5,2,3,2,4,2,3,2\}.$$

The details are shown in Figure 1.

Remark. We have found no way to predict the initial segment nor the central piece. Figure 2 shows the beginning of the case $l = 9$. From here one could be tempted to anticipate that this graph extends as in the case $l = 1$. This is not correct however, as can be seen in Figure 3. In fact, the initial segment is depicted in Figure 3 and its extension is shown in Figure 4.

The initial pattern can be quite elaborate. Figure 5 illustrates the case $l = 53$ and Figure 6 shows it for $l = 59$. A complete description of these initial segments is open to further exploration.

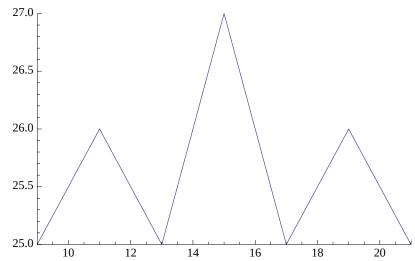

FIGURE 2. The beginning for $l = 9$

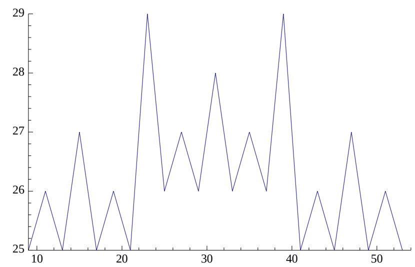

FIGURE 3. The continuation of $l = 9$

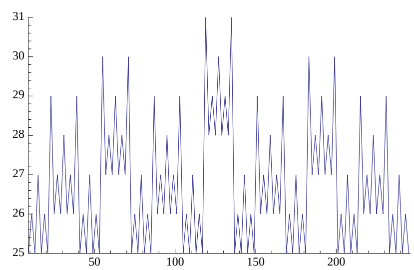

FIGURE 4. The pattern for $l = 9$ persists

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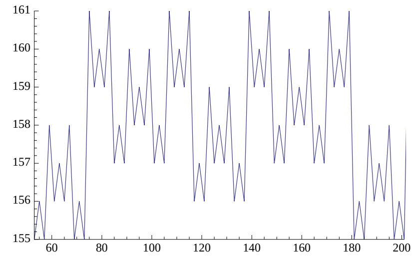

FIGURE 5. The initial pattern for $l = 53$

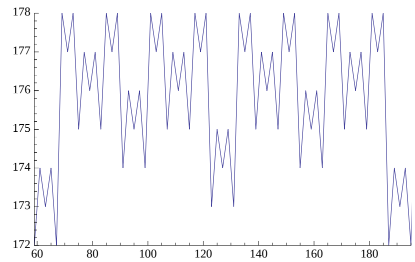

FIGURE 6. The initial pattern for $l = 59$

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