THE VALUES OF $\zeta(2n)$

In this note we use the expansion

$$\pi x \cot(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} B_{2n} \pi^{2n} x^{2n} \tag{1}$$

and the product representation

$$\sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right) \tag{2}$$

to obtain an expression for

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}}. \tag{3}$$

Taking logarithms in (2) gives

$$\ln \sin(\pi x) = \ln \pi + \ln x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2} \right) \tag{4}$$

and differentiation yields

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}. \tag{5}$$

Now use the geometric expansion

$$\frac{2x}{x^2 - n^2} = -\frac{2x}{n^2} \frac{1}{1 - x^2/n^2} = -2 \sum_{j=0}^{\infty} \frac{x^{2j+1}}{n^{2j+1}} \tag{6}$$

in (5) to produce

$$\begin{aligned} \pi \cot(\pi x) &= \frac{1}{x} - 2 \sum_{j=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2j+2}} \right) x^{2k+1} \\ &= \frac{1}{x} - 2 \sum_{j=1}^{\infty} \zeta(2j) x^{2j-1}. \end{aligned}$$

Therefore

$$\pi x \cot(\pi x) = 1 - 2 \sum_{j=1}^{\infty} \zeta(2j) x^{2j}. \tag{7}$$

Comparing with (1) gives

$$\zeta(2n) = (-1)^{n-1} B_{2n} \frac{2^{2n-1}}{(2n)!} \pi^{2n}. \tag{8}$$

Therefore $\zeta(2n)$ is a rational multiple of π^{2n}.

The formula implies that $(-1)^{n-1} B_{2n} > 0$. A direct proof of this result appears in [1].

In the expansion (7), it would be nice if the term "1" on the right, would correspond to the value in the sum with $j = 0$. This would require

$$\zeta(0) = -\frac{1}{2}. \tag{9}$$

This is indeed true, after the Riemann zeta function $\zeta(s)$ is extended in a proper form, to include $s = 0$ in its domain.

References

[1] L. J. Mordell. The sign of the Bernoulli numbers. *Amer. Math. Monthly*, 80:547–548, 1973.