RAABE'S THEOREM FOR BERNOULLI POLYNOMIALS

The Bernoulli polynomials have the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \tag{1}$$

The identity

$$B_k(x) + B_k\left(x + \tfrac{1}{2}\right) = 2^{1-k}B_k(2x), \qquad \text{for } k \geq 0, \tag{2}$$

was established in a previous note.

The generalization presented below is due to Raabe:

Theorem 1. *The Bernoulli polynomials satisfy the identity*

$$B_n(rx) = r^{n-1}\sum_{k=0}^{r-1} B_n\left(x + \frac{k}{r}\right), \qquad \text{for } r \geq 1 \text{ and } n \geq 0. \tag{3}$$

Compute the generating function of the right-hand side as

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{r-1} B_n\left(x + \frac{k}{r}\right)\frac{t^n}{n!} &= \sum_{k=0}^{r-1} \frac{t}{e^t - 1}e^{(x+k/r)t} \\ &= \frac{te^{xt}}{e^t - 1}\sum_{k=0}^{r-1} e^{kt/r} \\ &= \frac{te^{xt}}{e^t - 1} \times \frac{e^t - 1}{e^{t/r} - 1} \\ &= \frac{te^{xt}}{e^{t/r} - 1} \\ &= r\frac{se^{rxs}}{e^s - 1} \end{aligned} \tag{4}$$

with $s = t/r$. The last expression can be written as

$$r\frac{se^{rxs}}{e^s - 1} = \sum_{n=0}^{\infty} B_n(rx)\frac{s^n}{n!} \tag{5}$$

and comparing coefficients of t^n gives

$$\frac{1}{n!}\sum_{k=0}^{r-1} B_n\left(x + \frac{k}{r}\right) = \frac{r^{1-n}}{n!}B_n(rx). \tag{6}$$

This gives the result.

In the case $r = 3$ the identity gives

$$3^{1-n}B_n(3x) = B_n(x) + B_n\left(x + \tfrac{1}{3}\right) + B_n\left(x + \tfrac{2}{3}\right) \tag{7}$$

and letting $x = 0$ gives

$$\left(3^{1-n} - 1\right)B_n = B_n\left(\tfrac{1}{3}\right) + B_n\left(\tfrac{2}{3}\right). \tag{8}$$

The relation $B_n(1-x) = (-1)^n B_n(x)$ gives

$$B_n\left(\tfrac{2}{3}\right) = (-1)^n B_n\left(\tfrac{1}{3}\right) \tag{9}$$

and the using this on the right-hand side of (8) produces

$$\left(3^{1-n}-1\right) B_n = \left(1+(-1)^n\right) B_n\left(\tfrac{1}{3}\right). \tag{10}$$

This says nothing for n odd, but for n even it gives

$$B_n\left(\frac{1}{3}\right) = \frac{1}{2}(3^{1-n}-1)B_n \tag{11}$$

and, from (9),

$$B_n\left(\frac{2}{3}\right) = \frac{1}{2}(3^{1-n}-1)B_n \tag{12}$$

Theorem 2. *The Bernoulli polynomials satisfy*

$$B_{2n}\left(\frac{1}{3}\right) = B_{2n}\left(\frac{2}{3}\right) = \frac{1}{2}(3^{1-2n}-1)B_{2n}. \tag{13}$$

Question:What about the case n odd?

Now take $r=4$ in Raabe's theorem to obtain

$$4^{1-n}B_n(4x) = B_n(x) + B_n\left(x+\tfrac{1}{4}\right) + B_n\left(x+\tfrac{1}{2}\right) + B_n\left(x+\tfrac{3}{4}\right). \tag{14}$$

The special case $x=0$ gives

$$\left(4^{1-n}-1\right) B_n = B_n\left(\tfrac{1}{4}\right) + B_n\left(\tfrac{1}{2}\right) + B_n\left(\tfrac{3}{4}\right). \tag{15}$$

The use

$$B_n\left(\tfrac{1}{2}\right) = \left(2^{1-n}-1\right) B_n \tag{16}$$

and

$$B_n\left(\tfrac{3}{4}\right) = (-1)^n B_n\left(\tfrac{1}{4}\right) \tag{17}$$

to obtain

$$\left(2^{2-2n}-2^{1-n}\right) B_n = \left[1+(-1)^n\right] B_n\left(\tfrac{1}{4}\right). \tag{18}$$

The case n odd gives no information and the case n gives

$$B_{2n}\left(\tfrac{1}{4}\right) = \left(2^{1-4n}-2^{-2n}\right) B_{2n}. \tag{19}$$

Theorem 3. *The Bernoulli polynomials satisfy*

$$B_{2n}\left(\frac{1}{4}\right) = B_{2n}\left(\frac{3}{4}\right) = -\frac{2^{2n-1}-1}{2^{4n-1}}B_{2n}. \tag{20}$$