SCIENTIA
Series A: *Mathematical Sciences*, Vol. ?? (2015), ??
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The integrals in Gradshteyn and Ryzhik. Part 30: Trigonometric functions

Tewodros Amdeberhan, Atul Dixit, Xiao Guan, Lin Jiu, Alexey Kuznetsov, Victor H. Moll, and Christophe Vignat

Abstract. The table of Gradshteyn and Ryzhik contains many integrals that involve trigonometric functions. Evaluations are presented for integrands containing products of trigonometric functions and products of trigonometric functions and Legendre polynomials, logarithms, Bessel functions, and the Gauss hypergeometric function.

1. Introduction

This work forms part of the collection initiated in [**21**] with the goal of providing proofs and contact of the entries in the table of integrals [**12**]. As usual, the evaluations presented have a pedagogical component. The reader will find in this collection several proofs of the same result, as well as problems that appear in the process of writing the proofs. The authors consider important to discuss different approaches to these problems.

The table of integrals [**12**] contains a large class of entries where the integrand has a trigonometric part. These functions form part of the class of elementary functions, so it is natural that integrals involving them have been considered in detail. The goal of this note is to provide a sample of entries in [**12**] where the integrand is a combination of a basic trigonometric functions and a variety of other special functions.

The results of evaluations of integrals of elementary functions can be particularly beautiful. Moreover the arguments used in the proofs might not be self-evident. For instance, entry **4.229.7** is

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = \frac{\pi}{2} \ln \left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{2\pi} \right). \tag{1.1}$$

It is remarkable that the evaluation of this entry uses the so-called L-functions as described in [**29**]. A collection of integrals similar to (1.1) are given in [**5**] and [**17**]. A new method to evaluate such integrals has been given recently in [**8**].

2000 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals, trigonometric function.

2. Completely elementary entries

The most elementary examples appear in Section **2.01**, called `The basic integrals` as entries **2.01.5** and **2.01.6**

$$\int \sin x\, dx = -\cos x \quad \text{and} \quad \int \cos x\, dx = \sin x. \tag{2.1}$$

This section also contains the elementary evaluations **2.01.7** and **2.01.8**

$$\int \frac{dx}{\sin^2 x} = -\cot x \quad \text{and} \quad \int \frac{dx}{\cos^2 x} = \tan x \tag{2.2}$$

as well as

$$\int \frac{\sin x\, dx}{\cos^2 x} = \sec x \quad \text{and} \quad \int \frac{\cos x\, dx}{\sin^2 x} = -\text{cosec } x, \tag{2.3}$$

appearing as entries **2.01.9** and **2.01.10**, respectively. The final examples of trigonometric entries in this section are **2.01.11** and **2.01.12**

$$\int \tan x\, dx = -\ln \cos x \quad \text{and} \quad \int \cot x\, dx = \ln \sin x, \tag{2.4}$$

and also

$$\int \frac{dx}{\sin x} = \ln \tan \frac{x}{2} \quad \text{and} \quad \int \frac{dx}{\cos x} = \ln(\sec x + \tan x), \tag{2.5}$$

which appear as **2.01.13** and **2.01.14**, respectively.

3. Pure powers of sine and cosine

This section contains some explicit expressions for indefinite integrals of the form

$$I_{p,q}(x) = \int \sin^p x \, \cos^q x \, dx. \tag{3.1}$$

The first procedure to generate these evaluations comes from basic identities of trigonometric functions. The first result appears as entry **1.320.1** in [**12**].

Lemma 3.1. For $n \in \mathbb{N}$

$$\sin^{2n} x = \frac{1}{2^{2n}} \left\{ 2 \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \cos\left[2(n-k)x\right] + \binom{2n}{n} \right\}. \tag{3.2}$$

Proof. Start with the expansion

$$\sin^{2n} x = \left[\frac{e^{ix} - e^{-ix}}{2i} \right]^{2n} = \frac{1}{2^{2n}} \sum_{j=0}^{2n} (-1)^{n-j} \binom{2n}{j} e^{2ix(n-j)}. \tag{3.3}$$

The result follows by taking the real part and splitting the sum along $0 \leqslant j \leqslant n-1$, the term $j = n$ and then $n+1 \leqslant j \leqslant 2n$. □

Integrating the identity (3.2) gives entry **2.513.1**

$$\int \sin^{2n} x\, dx = \frac{x}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}. \tag{3.4}$$

The special definite integral

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{\pi}{2^{2n+1}} \binom{2n}{n}, \tag{3.5}$$

known as *Wallis' formula*, is now a direct consequence of (3.4). This appears as entry **3.621.3**, written in the semi-factorial notation

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}. \tag{3.6}$$

Similar identities are stated next. The proofs are omitted.

Lemma 3.2. For $n \in \mathbb{N}$, the identity

$$\sin^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n+k} \binom{2n+1}{k} \sin\left[(2n-2k+1)x\right] \tag{3.7}$$

holds. This appears as entry **1.320.3**. Integration yields

$$\int \sin^{2n+1} x\, dx = \frac{(-1)^{n+1}}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k} \tag{3.8}$$

that appears as entry **2.513.2** and integration gives

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n} \frac{(-1)^k \binom{2n+1}{k}}{2n+1-2k}. \tag{3.9}$$

The right-hand side of (3.9) can be reduced to the form stated in entry **3.621.4**:

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2^{2n}\, n!^2}{(2n+1)!}. \tag{3.10}$$

This is a typical question faced in the process of evaluating definite integrals. A procedure yields a form of the answer, usually in the form of a finite sum, and then it is required to match this to the one stated in [**12**]. This is illustrated next.

Lemma 3.3. For $n \in \mathbb{N}$, the identity

$$\frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n} \frac{(-1)^k \binom{2n+1}{k}}{2n+1-2k} = \frac{2^{2n}\, n!^2}{(2n+1)!} \tag{3.11}$$

holds.

PROOF. Write (3.11) in the form

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{2n+1}{k}}{2n-2k+1} = \frac{(-1)^n 2^{4n} n!^2}{(2n+1)!}.$$

This is now established by checking that both sides satisfy the same recurrence and that the initial conditions match. The recurrence is obtained from the Sigma package developed by C. Schneider in [**28**]. The output is that the left-hand side satisfies

$$8(n+1)f(n) + (2n+3)f(n+1) = 0. \tag{3.12}$$

It is easy to check that the right-hand side of (3.11) also satisfies (3.12), with the same initial conditions. The proof is complete. □

Lemma 3.4. For $n \in \mathbb{N}$, the identity

$$\cos^{2n} x = \frac{1}{2^{2n}} \left\{ 2 \sum_{k=0}^{n-1} \binom{2n}{k} \cos\left[(2n-2k)x\right] + \binom{2n}{n} \right\} \tag{3.13}$$

holds. This appears as entry **1.320.5**. Integration yields

$$\int \cos^{2n} x \, dx = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin\left[2(n-k)x\right]}{n-k} + \binom{2n}{n} x \right\}. \tag{3.14}$$

This appears as entry **2.513.3**. Integration gives entry **3.621.3**

$$\int_0^{\pi/2} \cos^{2n} x \, dx = \frac{\pi}{2^{2n+1}} \binom{2n}{n}. \tag{3.15}$$

Naturally this also follows from (3.5) by the change of variable $x \mapsto \frac{\pi}{2} - x$.

Lemma 3.5. For $n \in \mathbb{N}$, the identity

$$\cos^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos\left[(2n-2k+1)x\right] \tag{3.16}$$

holds. This appears as entry **1.320.7**. Integration yields entry **2.513.4**

$$\int \cos^{2n+1} x \, dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{\sin\left[(2n-2k+1)x\right]}{(2n-2k+1)}. \tag{3.17}$$

The change of variables $x \mapsto \frac{\pi}{2} - x$ gives

$$\int_0^{\pi/2} \cos^{2n+1} x \, dx = \int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2^{2n} n!^2}{(2n+1)!} \tag{3.18}$$

from **3.621.4** established in (3.10) and given in the table in the form $(2n)!!/(2n+1)!!$.

4. A first example

This section presents a proof of the evaluation stated as entry **3.631.16**.

Proposition 4.1. For $n \in \mathbb{N}$, the identity

$$\int_0^{\pi/2} \cos^n x \, \sin nx \, dx = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k} \tag{4.1}$$

holds.

Proof. The first proof uses the reduction formulas given in Section 3. Assume n is even, say $n = 2m$. The case n odd is treated by similar arguments. Start with the identity

$$\cos^{2m} x = \frac{1}{2^{2m}} \left[2 \sum_{k=0}^{m-1} \binom{2m}{k} \cos((2m-2k)x) + \binom{2m}{m} \right] \tag{4.2}$$

and the elementary evaluation

$$\int_0^{\pi/2} \cos(2jx) \sin(2kx)\, dx = \frac{\left[(-1)^{j+k} - 1\right] k}{2(j+k)(j-k)}, \quad \text{when } j \neq k, \tag{4.3}$$

to obtain, for $m \neq 0$,

$$\int_0^{\pi/2} \cos^{2m} x \, \sin 2mx \, dx = \frac{1}{2^{2m}} \left[\sum_{k=1}^{m-1} \left[1 - (-1)^k\right] \frac{n}{k(2m-k)} \binom{2m}{k} + \frac{1 - (-1)^m}{2m} \binom{2m}{m} \right].$$

The next lemma transforms the finite sum above into the form given in (4.1). This completes the proof. □

Lemma 4.2. For $m \in \mathbb{N}$, the identity

$$\sum_{k=1}^{m-1} \left[1 - (-1)^k\right] \frac{2m}{k(2m-k)} \binom{2m}{k} + \frac{1-(-1)^m}{m} \binom{2m}{m} = \sum_{k=1}^{2m} \frac{2^k}{k} \tag{4.4}$$

holds.

Proof. Observe that

$$\frac{2m}{k\,(2m-k)} = \frac{1}{k} + \frac{1}{2m-k}. \tag{4.5}$$

Then the left-hand side of (4.4) is

$$\begin{aligned} \text{LHS} &= \sum_{k=1}^{m-1} \left[1 - (-1)^k\right] \left(\frac{1}{k} + \frac{1}{2m-k} \right) \binom{2m}{k} + \frac{1-(-1)^m}{m} \binom{2m}{m} \\ &= \sum_{k=1}^{m-1} \left[1 - (-1)^k\right] \frac{1}{k} \binom{2m}{k} + \frac{1-(-1)^m}{m} \binom{2m}{m} + \sum_{k=1}^{m-1} \left[1 - (-1)^k\right] \frac{1}{2m-k} \binom{2m}{2m-k}. \end{aligned}$$

In the last sum, let $l = 2m - k$ to obtain

$$\begin{aligned} \text{LHS} &= \sum_{k=1}^{m-1} \left[1 - (-1)^k\right] \frac{1}{k} \binom{2m}{k} + \frac{1-(-1)^m}{m} \binom{2m}{m} + \sum_{l=m+1}^{2m-1} \left[1 - (-1)^{2m-l}\right] \frac{1}{l} \binom{2m}{l} \\ &= \sum_{k=1}^{2m-1} \left[1 - (-1)^k\right] \frac{1}{k} \binom{2m}{k}. \end{aligned}$$

The function $f(x) = \sum_{k=1}^{2m-1} \frac{1}{k}\binom{2m}{k} x^k$, satisfies LHS $= f(1) - f(-1)$. Then

$$f'(x) = \sum_{k=1}^{2m-1} \binom{2m}{k} x^{k-1} = \frac{(1+x)^{2m} - 1 - x^{2m}}{x} \tag{4.6}$$

gives

$$\text{LHS } = \int_{-1}^{1} f'(x)\, dx = \int_{-1}^{1} \frac{(1+x)^{2m} - 1}{x} dx = \sum_{k=1}^{2m} \frac{2^k}{k}.$$

The proof is complete. □

Note 4.3. The integral (4.1) can now be expressed in terms of the Chebyshev polynomials of the second kind. These are defined by the identity

$$U_r(\cos x) = \frac{\sin(r+1)x}{\sin x}. \tag{4.7}$$

Start with

$$\int_0^{\pi/2} \cos^n x \, \sin nx \, dx = \int_0^{\pi/2} \cos^n x \frac{\sin nx}{\sin x} \, \sin x \, dx \tag{4.8}$$

and make the change of variables $t = \cos x$ to obtain

$$\int_0^{\pi/2} \cos^n x \, \sin nx \, dx = \int_0^{1} t^n U_{n-1}(t)\, dt. \tag{4.9}$$

Integrals involving products of monomials and Chebyshev-U polynomials will be evaluated in Section 6. This will provide an alternative proof of Proposition 4.1.

5. Perhaps a related entry

Given enough patience, the reader will notice that there are pairs of entries in [**12**] with the same answer. For instance, entry **4.521.1** is

$$\int_0^1 \frac{\arcsin x}{x} \, dx = \frac{\pi}{2} \ln 2 \tag{5.1}$$

and entry **3.747.7** is

$$\int_0^{\pi/2} t \, \cot t \, dt = \frac{\pi}{2} \ln 2. \tag{5.2}$$

In this case, the change of variables $x = \sin t$ shows that both integrals are the same. The actual evaluation of these entries appears in [**2**].

It may be possible that the fact that two integrals have the same value, is simply a coincidence. This section discusses entry **3.274.2**. This appears in (5.3) and it agrees with entry **3.631.16** given in Proposition 4.1.

Proposition 5.1. For $n \in \mathbb{N}$, the identity

$$\int_0^1 \frac{1-x^n}{(1+x)^{n+1}} \frac{dx}{1-x} = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k} \tag{5.3}$$

holds.

Proof. Let $L(n)$ and $R(n)$ denote the left-hand side (right-hand side) of (5.3), respectively. It is shown that both expressions satisfy the difference equation

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{2n}, \qquad \text{with } x_1 = \tfrac{1}{2}. \tag{5.4}$$

Observe that

$$2^{n+1}R(n) = \sum_{k=1}^{n} \frac{2^k}{k} = \sum_{k=1}^{n-1} \frac{2^k}{k} + \frac{2^n}{n} = 2^n R(n-1) + \frac{2^n}{n},$$

showing that $R(n)$ satisfies the stated recurrence. For the function $L(n)$, compute

$$\begin{aligned} L(n) - \tfrac{1}{2}L(n-1) &= \int_0^1 \frac{(1-x^n)}{(1+x)^{n+1}\,(1-x)}\,dx - \int_0^1 \frac{(1-x^{n-1})}{2(1+x)^n\,(1-x)}\,dx \\ &= \int_0^1 \frac{1}{2(1+x)^{n+1}\,(1-x)} \left[2(1-x^n) - (1-x^{n-1})(1+x)\right]\,dx \\ &= \int_0^1 \frac{1}{2(1+x)^{n+1}\,(1-x)} \left[(1-x)(1+x^{n-1})\right]\,dx \\ &= \frac{1}{2}\int_0^1 \frac{1+x^{n-1}}{(1+x)^{n+1}}\,dx. \end{aligned}$$

The change of variables $t = 1/x$ shows that

$$\int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{n+1}} = \int_1^\infty \frac{dt}{(1+t)^{n+1}} \tag{5.5}$$

that produces

$$L(n) - \tfrac{1}{2}L(n-1) = \frac{1}{2}\int_0^\infty \frac{dx}{(1+x)^{n+1}} = \frac{1}{2n}. \tag{5.6}$$

The initial condition $L(1) = 1/2$ is elementary. Therefore $L(n)$ satisfies the same recurrence as $R(n)$, with the same initial condition. This completes the proof that $L(n) = R(n)$. □

The identity

$$\int_0^1 \frac{(1-x^n)\,dx}{(1+x)^{n+1}\,(1-x)} = \int_0^{\pi/2} \cos^n x\,\sin nx\,dx \tag{5.7}$$

is now established. An interpretation in terms of Chebyshev polynomials is presented next.

The matching of the sides of (5.7) is now written as an identity showing that two rational functions have the same integral. The first step is to transform (4.1) into a rational form.

Lemma 5.2. For $n \in \mathbb{N}$,

$$\int_0^{\pi/2} \cos^n x \, \sin nx \, dx = 2 \int_0^1 \frac{(1-v)^n}{(1+v)^{n+2}} U_{n-1}\left(\frac{1-v}{1+v}\right) dv. \tag{5.8}$$

Proof. Start with

$$\int_0^{\pi/2} \cos^n x \, \sin nx \, dx = \int_0^{\pi/2} \cos^n x \, \frac{\sin nx}{\sin x} \, \sin x \, dx \tag{5.9}$$

and use (4.7) followed by the Weierstrass change of variables $u = \tan x/2$ to obtain the result after the change of variable $v = u^2$. □

Theorem 5.3. Consider the two families of rational functions defined by

$$Y_{1,n}(x) = \frac{1-x^n}{(1+x)^{n+1}(1-x)} \tag{5.10}$$

and

$$Y_{2,n}(x) = \frac{2(1-x)^n}{(1+x)^{n+2}} U_{n-1}\left(\frac{1-x}{1+x}\right). \tag{5.11}$$

Then

$$\int_0^1 Y_{1,n}(x) \, dx = \int_0^1 Y_{2,n}(x) \, dx \tag{5.12}$$

for every $n \in \mathbb{N}$.

Note 5.4. It is an interesting question to prove the identity (5.12) by a direct change of variables. A non-systematic procedure, using `Mathematica`, shows that

$$x(t) = \frac{2t}{1+t^2} \tag{5.13}$$

gives

$$\int_0^1 \frac{dx}{(1+x)^2} = \int_0^1 \frac{2(1-t)\, dt}{(1+t)^3}, \tag{5.14}$$

and

$$x(t) = \frac{4t(1+t^2)}{t^4+6t^2+1} \tag{5.15}$$

gives

$$\int_0^1 \frac{dx}{(1+x)^2} = \int_0^1 \frac{4(1-t)^3\, dt}{(1+t)^5}. \tag{5.16}$$

These are the cases $n = 1$ and $n = 2$ in (5.12). The reader is encouraged to try the next case and find a change of variables to prove

$$\int_0^1 \frac{x^2+x+1}{(1+x)^4} \, dx = \int_0^1 \frac{2(1-t)^3(3t^2-10t+3)\, dt}{(1+t)^7}. \tag{5.17}$$

(The common value is $\frac{5}{12}$).

Note 5.5. The identity of Theorem 5.3 admits an automatic proof described next. Define

$$V_1(n) = \int_0^1 \frac{1-x^n}{(1+x)^{n+1}(1-x)}\, dx \tag{5.18}$$

and

$$V_2(n) = \int_0^1 \frac{2(1-x)^n}{(1+x)^{n+2}} U_{n-1}\left(\frac{1-x}{1+x}\right)\, dx. \tag{5.19}$$

The identity in Theorem 5.3 is equivalent to $V_1(n) = V_2(n)$, for all $n \in \mathbb{N}$.

The proof begins with the generating functions

$$G_1(x,t) = \sum_{k=0}^{\infty} \frac{1-x^k}{(1+x)^{k+1}(1-x)} t^k = \frac{t}{(t-x-1)(tx-x-1)}$$

and

$$\begin{aligned} G_2(x,t) &= \sum_{k=1}^{\infty} \frac{2(1-x)^k}{(1+x)^{k+2}} U_{k-1}\left(\frac{1-x}{1+x}\right) t^k \\ &= -\frac{2t(x-1)}{(x+1)(-2t(x-1)^2 + t^2(x-1)^2 + (x+1)^2)} \\ &= -\frac{2t(x-1)}{(x+1)(t^2x^2 - 2t^2x + t^2 - 2tx^2 + 4tx - 2t + x^2 + 2x + 1)}. \end{aligned}$$

This last expression follows from the generating function for the Chebyshev polynomials

$$\sum_{n=0}^{\infty} U_n(x) t^n = \frac{1}{1-2tx+t^2}. \tag{5.20}$$

Define

$$F_1(n|x,t) = \frac{G_1(x,t)}{t^{n+1}} \text{ and } F_2(n|x,t) = \frac{G_2(x,t)}{t^{n+1}}. \tag{5.21}$$

The function `MAZ` in the package `MultiAlmkvistZeilberger`, available in D. Zeilberger's website at

`http://www.math.rutgers.edu/~zeilberg/tokhniot/MultiAlmkvistZeilberger`

produces

$$\begin{aligned} F_1(n+1|x,t) - \frac{1}{2}F_1(n|x,t) &= \frac{d}{dx}\left(\frac{x^2-1}{2(n+1)t}\frac{G_1(x,t)}{t^{n+1}}\right) + \frac{d}{dt}\left(\frac{t-2}{2(n+1)}\frac{G_1(x,t)}{t^{n+1}}\right) \\ &= -\frac{t-2}{2(t-x-1)((t-1)x-1)t^{n+1}} \end{aligned}$$

and

$$\begin{aligned} F_2(n+2|x,t) - \frac{1}{2}F_2(n+1|x,t) &= \frac{d}{dx}\left(-\frac{tx^2-t-x^2+1}{4(n+2)t^2}\frac{G_2(x,t)}{t^{n+1}} \right) \\ &\quad + \frac{d}{dt}\left(\frac{t-2}{2(n+2)t}\frac{G_2(x,t)}{t^{n+1}} \right) \\ &= \frac{(t-2)(x-1)}{(x+1)\left(t^2(x-1)^2-2t(x-1)^2+(x+1)^2\right)t^{n+2}}. \end{aligned}$$

Multiplying the first relation by t^{n+2} and integrating from $x=0$ to 1 yields

$$\begin{aligned} \sum_{n=1}^{\infty} \left(V_1(n) - \tfrac{1}{2}V_1(n-1) \right) t^n &= -\int_0^1 \frac{(t-2)t\,dx}{2(t-x-1)((t-1)x-1)} \\ &= -\frac{1}{2}\log(1-t). \end{aligned} \tag{5.22}$$

Similarly, multiplying the second relation above by t^{n+3} and integrating from $x=0$ to 1 gives

$$\begin{aligned} \sum_{n=1}^{\infty} \left(V_2(n) - \tfrac{1}{2}V_2(n-1) \right) t^n &= \int_0^1 \frac{t(t-2)(x-1)\,dx}{(x+1)\left(t^2(x-1)^2-2t(x-1)^2+(x+1)^2\right)} \\ &= -\frac{1}{2}\log(1-t). \end{aligned}$$

It follows from here that $V_1(n)$ and $V_2(n)$ satisfy the same first order recurrence. The identity $V_1(n) \equiv V_2(n)$ now follows from the fact that this initial conditions match. Indeed,

$$V_1(1) = \int_0^1 \frac{dx}{(1+x)^2} = \frac{1}{2} \tag{5.23}$$

and

$$V_2(1) = \int_0^1 \frac{2(1-x)\,dx}{(1+x)^3} = \frac{1}{2}. \tag{5.24}$$

6. A family of integrals involving Chebyshev-U polynomials

This section provides closed-form expressions for the family of integrals

$$I_{j,n} = \int_0^1 x^j U_n(x)\,dx, \tag{6.1}$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind defined in (4.7). The example stated in Proposition 4.1 is

$$I_{n,n-1} = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k}. \tag{6.2}$$

An experimental search in `Mathematica` shows that

$$I_{n,n+1} = \int_0^1 t^n U_{n+1}(t)\,dt = \frac{1}{n} \tag{6.3}$$

and

$$I_{n,n+3} = \int_0^1 t^n U_{n+3}(t)\, dt = \frac{n}{(n+1)(n+2)}. \tag{6.4}$$

The search for closed-form expressions for $I_{j,n}$ was partially motivated by seeking an explanation of the simplicity of these forms.

Theorem 6.1. The integral $I_{j,n}$ is given by

$$I_{j,n} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \frac{2^{n-2k}}{n+j+1-2k}. \tag{6.5}$$

Proof. This follows directly from the expression

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \tag{6.6}$$

that appears in [**1**, 22.3.7, p. 775]. □

Note 6.2. The expression (6.6) can be written as

$$U_n(x) = \sum_{k=0}^{n-1} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \tag{6.7}$$

and it yields

$$I_{j,n} = \sum_{k=0}^{n-1} (-1)^k \binom{n-k}{k} \frac{2^{n-2k}}{n+j+1-2k}. \tag{6.8}$$

since the extra terms, namely those with $k > \lfloor \frac{n}{2} \rfloor$, vanish. Observe that the vanishing of $n+j+1-2k$ requires $n+j$ to be odd, say, $n+j = 2r-1$. Then $k = r = \frac{1}{2}(n+j+1)$ and this occurs only when $r < n-1$; that is, when $j < n-3$. In such case, the corresponding binomial coefficient is $\binom{n-k}{k} = \binom{r-1-j}{r} = 0$.

Example 6.3. The case considered in Proposition 4.1 has the alternative expression

$$I_{n,n-1} = \sum_{k=0}^{n-2} (-1)^k \binom{n-1-k}{k} \frac{2^{n-2-2k}}{n-k}, \quad \text{for } n \geqslant 2. \tag{6.9}$$

The proof of this identity uses recurrences produced by the Sigma package developed in [**28**]. The output of this package is that left-hand side satisfies the recurrence

$$\mathcal{L}(f(n)) = (n+1)f(n) - (3n+4)f(n+1) + (2n+4)f(n+2) = 0, \tag{6.10}$$

where $f(n)$ is the left-hand side of (6.9). It is now shown that

$$g(n) = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k} \tag{6.11}$$

satisfies the same recurrence. Indeed,

$$\begin{aligned}
\mathcal{L}(g(n)) &= (n+1)g(n) + (2n+4)g(n+2) - (3n+4)g(n+1) \\
&= \frac{n+1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k} + \frac{2n+4}{2^{n+3}} \sum_{k=1}^{n+2} \frac{2^k}{k} - \frac{3n+4}{2^{n+2}} \sum_{k=1}^{n+1} \frac{2^k}{k} \\
&= \frac{4n+4+2n+4-6n-8}{2^{n+3}} \sum_{k=1}^{n} \frac{2^k}{k} + \frac{1}{4}\left(\frac{2n+4-6n-8}{n+1} + 4 \right) \\
&= 0.
\end{aligned}$$

The result now follows by verifying that $f(n)$ and $g(n)$ have the same two initial values.

6.1. An alternative proof. A new closed-form for the integrals $I_{j,n}$ is presented next. The analysis begins with the Fourier transform of the Chebyshev polynomial

$$\widehat{U}_n(\omega) = \int_{-1}^{1} U_n(x) e^{i\omega x}\, dx \tag{6.12}$$

and the expression

$$\begin{aligned}
\widehat{U}_n(\omega) &= \sum_{k=0}^{n} \frac{2^{2k+1}(n+k+1)!k!}{(2k+1)!(n-k)!} \frac{\left[(-1)^{n-k}e^{-i\omega} - e^{i\omega}\right]}{(-2i\omega)^{k+1}} \\
&= \sum_{k=0}^{n} 2^k k! \binom{n+k+1}{2k+1} i^{k+1} \frac{\left[(-1)^{n-k}e^{-i\omega} - e^{i\omega}\right]}{\omega^{k+1}}
\end{aligned} \tag{6.13}$$

provided in [**9**].

Proposition 6.4. Assume $n+j$ is even. Then

$$I_{j,n} = \int_0^1 t^j U_n(t)\, dt = \frac{2}{i^j} \left(\frac{d}{d\omega} \right)^j \widehat{U}_n(\omega) \Big|_{\omega=0}. \tag{6.14}$$

Proof. Use $U_n(-x) = (-1)^n U_n(x)$ to obtain

$$\widehat{U}_n(\omega) = \int_0^1 \left[e^{i\omega x} + (-1)^n e^{-i\omega x} \right] U_n(x)\, dx. \tag{6.15}$$

Differentiating j times with respect to ω gives

$$\int_0^1 (ix)^j \left[e^{i\omega x} + (-1)^{n+j} e^{-i\omega x} \right] U_n(x)\, dx = \left(\frac{d}{d\omega} \right)^j \widehat{U}_n(\omega). \tag{6.16}$$

Replacing $\omega = 0$ gives the result. □

The next step is to provide a direct proof that $\widehat{U}_n(\omega)$ is an analytic function of ω.

Proposition 6.5. The Fourier transform of the Chebyshev polynomial $U_n(x)$ is given by

$$\widehat{U}_n(\omega) = \sum_{k=0}^{n} 2^k k! \binom{n+k+1}{2k+1} i^{k+1} \sum_{r=k+1}^{\infty} \frac{i^r \left[(-1)^{n-k-r} - 1 \right]}{r!} \omega^{r-k-1}. \tag{6.17}$$

Proof. Expanding the exponential functions in (6.13) gives

$$\widehat{U}_n(\omega) = \sum_{k=0}^{n} 2^k k! \binom{n+k+1}{2k+1} i^{k+1} \sum_{r=0}^{\infty} \frac{i^r \left[(-1)^{n-k-r} - 1\right]}{r!} \omega^{r-k-1}. \tag{6.18}$$

It remains to show that the negative powers of ω vanish. The contribution of those negative powers is

$$S_n(\omega) = \sum_{k=0}^{n} 2^k k! \binom{n+k+1}{2k+1} i^{k+1} \sum_{r=0}^{k} \frac{i^r \left[(-1)^{n-k-r} - 1\right]}{r!} \omega^{r-k-1}. \tag{6.19}$$

Exchanging the order of summation gives

$$S_n(\omega) = \sum_{\ell=-n-1}^{-1} i^{\ell} \left[(-1)^{n-\ell-1} - 1\right] \left(\sum_{k=-\ell-1}^{n} \frac{(-1)^{k+1} 2^k k! \binom{n+k+1}{2k+1}}{(\ell+k+1)!} \right) \omega^{\ell}. \tag{6.20}$$

The coefficient of ω^ℓ vanishes if n and ℓ are of opposite parity. The case of same parity is given in the lemma below. This completes the proof. □

Lemma 6.6. Let n, ℓ be positive integers with the same parity. Then

$$\sum_{k=-\ell-1}^{n} \frac{(-1)^k 2^k k!}{(\ell+k+1)!} \binom{n+k+1}{2k+1} = 0 \tag{6.21}$$

for $-n-1 \leqslant \ell \leqslant -1$.

Proof. The proof is based on the WZ-method. A nice description of this procedure may be found in [**22**]. Let $m = 1 - \ell$ so that $0 \leqslant m \leqslant n$ and the identity in question reads (m, n opposing parity)

$$\sum_{k=m}^{n} \frac{(-1)^k 2^k k!}{(k-m)!} \binom{n+k+1}{2k+1} = m! \sum_{k=m}^{n} (-2)^k \binom{k}{m} \binom{n+k+1}{2k+1} = 0.$$

Case 1. ($n \to 2n, m \to 2m-1$). Define the sum $A(n;m) := W(n;m) \sum_{k=2m-1}^{2n} F(n,k;m)$ where

$$W(n;m) := \frac{(-1)^n \binom{n+m-1}{n-m}}{\binom{2n+2m-1}{2n-2m+1}\binom{4m-2}{2m-1}} \qquad \text{and} \qquad F(n,k;m) := (-2)^k \binom{k}{2m-1} \binom{2n+k+1}{2n-k}.$$

The WZ algorithm generates

$$A(n+1;m) - A(n;m) = \sum_k G(n,k+1;m) - \sum_k G(n,k;m) = 0 \tag{6.22}$$

for

$$G(n,k;m) := \frac{4(n+1)(k-2m+1)(2k+1)W(n;m)F(n,k;m)}{(2n-k+1)(2n-k+2)(2n+2m+1)}.$$

Checking initial condition, say $A(m;m) = 0$, proves the assertion.

Case 2. $(n \to 2n-1, m \to 2m)$. Define the sum $A(n;m) := W(n;m)\sum_{k=2m}^{2n-1} F(n,k;m)$ where

$$W(n;m) := \frac{(-1)^n \binom{n+m-1}{n-m-1}}{\binom{2n+2m-1}{2n-2m-1}\binom{4m}{2m}} \qquad \text{and} \qquad F(n,k;m) := (-2)^k \binom{k}{2m}\binom{2n+k}{2n-1-k}.$$

The WZ algorithm generates

$$A(n+1;m) - A(n;m) = \sum_k G(n,k+1;m) - \sum_k G(n,k;m) = 0 \tag{6.23}$$

for

$$G(n,k;m) := \frac{2(2n+1)(k-2m)(2k+1)W(n;m)F(n,k;m)}{(2n-k)(2n-k+1)(2n+2m+1)}.$$

Checking initial condition, say $A(m+1;m) = 0$, proves the assertion. □

Lemma 6.6 shows that the Fourier transform of the Chebyshev-U polynomials can be written as

$$\widehat{U}_n(\omega) = \sum_{k=0}^{n} (-1)^{k+1} 2^k k! \binom{n+k+1}{2k+1} \sum_{s=0}^{\infty} \frac{i^s \left[(-1)^{n-1-s} - 1\right]}{(s+k+1)!}\omega^s \tag{6.24}$$

and differentiating j times gives

$$\left(\frac{d}{d\omega}\right)^j \widehat{U}_n(\omega) = \tag{6.25}$$
$$\sum_{k=0}^{n} (-1)^{k+1} 2^k k! \binom{n+k+1}{2k+1} \sum_{s=j}^{\infty} s(s-1)\cdots(s-j+1)\frac{i^s \left[(-1)^{n-1-s} - 1\right]}{(s+k+1)!}\omega^{s-j}.$$

In order to use the result of Proposition 6.4, compute the derivative at $\omega = 0$ to obtain

$$\left(\frac{d}{d\omega}\right)^j \widehat{U}_n(\omega)\Big|_{\omega=0} = \sum_{k=0}^{n} (-1)^{k+1} 2^k k! \binom{n+k+1}{2k+1} j! i^j \frac{\left[(-1)^{n-1-j} - 1\right]}{(k+j+1)!}. \tag{6.26}$$

In the case $n+j$ even, this yields

$$\frac{2}{i^j}\left(\frac{d}{d\omega}\right)^j \widehat{U}_n(\omega)\Big|_{\omega=0} = 2\sum_{k=0}^{n} (-1)^k 2^k \binom{n+k+1}{2k+1} \frac{j!\,k!}{(k+j+1)!}. \tag{6.27}$$

The previous arguments evaluate the integral $I_{j,n}$ in the case j and n have the same parity.

Proposition 6.7. Assume $n+j$ is even. Then

$$I_{j,n} = 2\sum_{k=0}^{n} (-1)^k 2^k \binom{n+k+1}{2k+1} \frac{j!\,k!}{(k+j+1)!}. \tag{6.28}$$

7. Integrals expressed in terms of the digamma function

This section discusses two entries in [**12**] where the integrand has a trigonometric function and the value is given in terms of the digamma function

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{7.1}$$

This function admits a variety of integral representations, starting with entry **8.361.7**

$$\psi(x) = \int_0^1 \frac{t^{x-1} - 1}{t - 1} \, dt - \gamma \tag{7.2}$$

with $\gamma = -\Gamma'(1)$ being Euler's constant. These representations are established in [**17**].

Example 7.1. Entry **3.688.1** is

$$\int_0^{\pi/4} \frac{\tan^\nu x - \tan^\mu x}{\cos x - \sin x} \frac{dx}{\sin x} = \psi(\mu) - \psi(\nu). \tag{7.3}$$

This is evaluated by writing it as

$$I = \int_0^{\pi/4} \frac{\tan^{\nu-1} x - \tan^{\mu-1} x}{1 - \tan x} \frac{dx}{\cos^2 x} \tag{7.4}$$

and transforming it, by the change of variables $s = \tan x$, into

$$I = \int_0^1 \frac{s^{\nu-1} - s^{\mu-1}}{1 - s} \, ds. \tag{7.5}$$

This integral appears as entry **3.231.5** with value $\psi(\mu) - \psi(\nu)$. This entry was established as Proposition 3.1 in [**18**].

Example 7.2. Entry **3.624.6** is

$$\int_0^{\pi/2} \left(\frac{\sin ax}{\sin x} \right)^2 dx = \frac{\pi a}{2} - \frac{1}{2} \sin \pi a \, [2a\beta(a) - 1] \tag{7.6}$$

where

$$\beta(x) = \frac{1}{2} \left[\psi\left(\frac{x+1}{2} \right) - \psi\left(\frac{x}{2} \right) \right] \tag{7.7}$$

is defined in entry **8.370**.

Proof of a special case. The proof is presented first in the case $a \in \mathbb{N}$. In this special case the formula becomes

$$\int_0^{\pi/2} \left(\frac{\sin ax}{\sin x} \right)^2 dx = \frac{\pi a}{2} \tag{7.8}$$

since the factor $\sin \pi a$ vanishes and $\beta(a)$ has a finite value.

Recall the Fejer kernel [**14**]

$$F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x) = \frac{1}{n} \left(\frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \right)^2 \tag{7.9}$$

with

$$D_k(x) = \sum_{\ell=-k}^{k} e^{i\ell x} \tag{7.10}$$

satisfies

$$\int_0^\pi F_n(x)\,dx = \pi. \tag{7.11}$$

The change of variables $x = 2t$ and replacing n by a in (7.11) gives (7.6).

A proof by contour integration. The function $f(z) = (\sin(az)/\sin z)^2$ is analytic inside the rectangle with vertices $P_1 = (0,0)$, $P_2 = (\pi/2, 0)$, $P_3 = (\pi/2, B)$ and $P_4 = (0, B)$. Integration produces

$$0 = \int_L f(z)dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{L_3} f(z)dz + \int_{L_4} f(z)dz, \tag{7.12}$$

where L_j is the segment joining P_{j-1} to P_j (with $P_4 = P_0$). The first integral gives the left-hand side of (7.8). In the second integral, observe that the integrand is

$$\int_{L_2} f(z)dz = i\int_0^B \frac{\sin^2\left[a(\pi/2+it)\right]}{\cosh^2 t}\,dt. \tag{7.13}$$

The integrand is of order $e^{-2(1-a)t}$ and letting $B \to \infty$ gives

$$\int_{L_2} f(z)dz = i\int_0^\infty \frac{\sin^2\left[a(\pi/2+it)\right]}{\cosh^2 t}\,dt. \tag{7.14}$$

The identity $\sin^2 u = \frac{1}{2}(1-\cos(2u))$ gives

$$\int_{L_2} f(z)dz = \frac{i}{2} - \frac{i\cos(\pi a)}{2}\int_0^\infty \frac{\cosh(2at)}{\cosh^2 t}\,dt - \frac{\sin\pi a}{2}\int_0^\infty \frac{\sinh(2at)}{\cosh^2 t}\,dt. \tag{7.15}$$

A similar argument gives

$$\int_{L_4} f(z)\,dz = -i\int_0^\infty \frac{\sinh^2(ax)}{\sinh^2 x}\,dx \tag{7.16}$$

and the integral over L_3 vanishes as $B \to \infty$. The real part of (7.12) now gives

$$\int_0^\infty \left(\frac{\sin ax}{\sin x}\right)^2 dx = \frac{\sin(\pi a)}{2}\int_0^\infty \frac{\sinh(2ax)\,dx}{\cosh^2 x}. \tag{7.17}$$

Now use entry 3.541.8

$$\int_0^\infty \frac{e^{-ax}\,dx}{\cosh^2 x} = a\beta\left(\frac{a}{2}\right) - 1 \tag{7.18}$$

(which appears incorrectly in the latest edition [**12**] and **correctly** in [**11**], an error caused by the sixth author of this note) it follows that

$$\int_0^\infty \left(\frac{\sin ax}{\sin x}\right)^2 dx = -\frac{a\sin\pi a}{2}\left(\beta(a)+\beta(-a)\right). \tag{7.19}$$

A proof of (7.18) may be found in [**6**]. The result now follows from the identity

$$\beta(a) + \beta(-a) = 2\beta(a) - \frac{\pi}{\sin \pi a} - \frac{1}{a} \tag{7.20}$$

that can be verified directly from the definition of $\beta(x)$ and the elementary properties

$$\psi(x+1) = \psi(x) + \frac{1}{x} \text{ and } \psi(1-x) = \psi(x) + \pi \cot \pi x. \tag{7.21}$$

8. Integrals expressed in terms of Legendre polynomials

A variety of entries in [**12**] involve trigonometric functions in the integrand and the result is given in terms of the Legendre polynomials $P_n(x)$ defined by the Rodrigues' formula

$$P_n(x) = \frac{1}{2^n \, n!} \frac{d^n}{dx^n} (x^2-1)^n \tag{8.1}$$

(appearing as entry **8.910.2**), which have the explicit formula

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} \tag{8.2}$$

(this is entry **8.911.1**) and generating function

$$\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}}. \tag{8.3}$$

A selection of them is presented here. The next section contains some entries in [**12**] where the integrand is a combination of Legendre polynomials and trigonometric functions. Legendre polynomials form an orthogonal sequence on the interval $[-1,1]$ with respect to the measure $d\mu(x) = \mathbf{1}_{[-1,1]}(x) \, dx$ and normalization factor

$$\int_{-1}^{1} P_n^2(x) \, dx = \frac{2}{2n+1}. \tag{8.4}$$

Sequences of orthogonal of polynomials are characterized by a three-term recurrence. In this case, this is

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{8.5}$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x$.

Example 8.1. The first formula is half of Entry **3.611.3** and it is the classical *Laplace-Mehler integral*:

$$P_n(\cos\theta) = \frac{1}{\pi} \int_0^{\pi} (\cos\theta + i \sin\theta \cos\varphi)^n d\varphi = \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta + i \sin\theta \cos\varphi)^n d\varphi. \tag{8.6}$$

To prove this formula, let $x = \cos\theta$ and define

$$M_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x + i\sqrt{1-x^2} \cos\varphi \right]^n \, d\varphi. \tag{8.7}$$

Expanding the n-th power and using (3.5) and the fact that odd powers of cosine have vanishing integral, gives

$$M_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2^{2k}} \binom{n}{2k} \binom{2k}{k} x^{n-2k} (1-x^2)^k. \tag{8.8}$$

Expand the term $(1-x^2)^k$ and reverse the order of summation to obtain

$$M_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \left[\sum_{k=r}^{\lfloor n/2 \rfloor} 2^{-2k} \binom{n}{2k} \binom{2k}{k} \binom{k}{r} \right] (-1)^r x^{n-2r}. \tag{8.9}$$

The proof that $M_n(x)$ is the Legendre polynomial in (8.2) is equivalent to the identity

$$\sum_{k=r}^{\lfloor n/2 \rfloor} 2^{-2k} \binom{n}{2k} \binom{2k}{k} \binom{k}{r} = \frac{(2n-2r)!}{2^n r! (n-r)! (n-2r)!}, \text{ for } 0 \leqslant r \leqslant \lfloor n/2 \rfloor . \tag{8.10}$$

Separating this sum according to the parity shows the next result.

Lemma 8.2. Assume the identities

$$\sum_{j=0}^{m} \frac{2^{2(m-j)} (2m+2r)! (2m+r)! (2m)!}{(2m-2j)! (j+r)! j! (4m+2r)!} = 1 \tag{8.11}$$

and

$$\sum_{j=0}^{m} \frac{2^{2(m-j)+1} (2m+2r+1)! (2m+r+1)! (2m+1)!}{(2m+1-2j)! (j+r)! j! (4m+2r+2)!} = 1 \tag{8.12}$$

hold for all $m \in \mathbb{N}$ and all $0 \leqslant r \leqslant m$. This implies the Laplace-Mehler representation (8.6) for Legendre polynomials.

It remains to confirm the assumptions of Lemma 8.2. Denote by $f_1(m)$ and $f_2(m)$, respectively, the sums appearing in (8.11) and (8.12). The Sigma package developed by [**28**] produces the trivial recurrence

$$f_j(m+1) = f_j(m), \quad \text{for } j = 1, 2 \text{ and } m \in \mathbb{N}. \tag{8.13}$$

The initial values $f_1(1) = f_2(1) = 1$ confirms that $f_1(m) \equiv f_2(m) \equiv 1$. This completes the proof of (8.6).

Example 8.3. Entry **3.661.3** is

$$\int_0^\pi (a + b\cos x)^n \, dx = \pi (a^2 - b^2)^{n/2} P_n \left(\frac{a}{\sqrt{a^2 - b^2}} \right), \quad \text{for } |a| > |b|. \tag{8.14}$$

This evaluation uses the representation

$$P_n(\cosh\theta) = \frac{1}{\pi} \int_0^\pi (\cosh\theta - \sinh\theta \cos\varphi)^n \, d\varphi \tag{8.15}$$

obtained by replacing θ by $i\theta$ in the Laplace-Mehler representation. To obtain the result, write

$$\int_0^\pi (a + b\cos x)^n \, dx = (a^2 - b^2)^{n/2} \int_0^\pi \left[\frac{a}{\sqrt{a^2 - b^2}} - \frac{(-b)}{\sqrt{a^2 - b^2}} \cos\varphi \right]^n d\varphi, \tag{8.16}$$

and choose the angle θ so that

$$\cosh\theta = \frac{a}{\sqrt{a^2-b^2}}. \tag{8.17}$$

This is possible since $a \geqslant \sqrt{a^2-b^2}$. This proves the statement.

Example 8.4. The entry **3.611.4**

$$\int_0^\pi \frac{dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{(a^2-b^2)^{(n+1)/2}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right) \tag{8.18}$$

is a companion to Entry **3.661.3** established in the previous example. Introduce the parameter θ by the relation

$$\cosh\theta = \frac{a}{\sqrt{a^2-b^2}} \text{ and } \sinh\theta = \frac{b}{\sqrt{a^2-b^2}}. \tag{8.19}$$

Then (8.18) is written as

$$P_n(\cosh\theta) = \frac{1}{\pi}\int_0^\pi \frac{du}{(\cosh\theta+\sinh\theta\cos u)^{n+1}}. \tag{8.20}$$

To prove this identity, it is shown that the generating functions of both sides agree. For the left-hand side, this is

$$\sum_{n=0}^\infty P_n(\cosh\theta)\, t^n = \frac{1}{\sqrt{1-2t\,\cosh\theta+t^2}}. \tag{8.21}$$

On the other hand, for the right-hand side this generating function is

$$\frac{1}{\pi t}\int_0^\pi \sum_{n=0}^\infty \left[\frac{t}{\cosh\theta+\sinh\theta\cos u}\right]^{n+1} du \quad = \quad \frac{1}{\pi}\int_0^\pi \frac{du}{(\cosh\theta - t)+\sinh\theta\cos u}.$$

The result now follows from the elementary integral

$$\int_0^\pi \frac{du}{a+b\cos u} = \frac{\pi}{\sqrt{a^2-b^2}}, \quad \text{for } |a|>|b|. \tag{8.22}$$

Example 8.5. Entry **3.675.1** is

$$\int_u^\pi \frac{\sin(n+\frac{1}{2})x\,dx}{\sqrt{2(\cos u-\cos x)}} = \frac{\pi}{2}P_n(\cos u) \tag{8.23}$$

and its companion **3.675.2**

$$\int_0^u \frac{\cos(n+\frac{1}{2})x\,dx}{\sqrt{2(\cos x-\cos u)}} = \frac{\pi}{2}P_n(\cos u) \tag{8.24}$$

This will be established by computing the generating function of both sides and verifying that they agree. It is convenient to change the names of the variables and write (8.23) as

$$P_n(\cos u) = \frac{2}{\pi}\int_u^\pi \frac{\sin(n+\frac{1}{2})w\,dw}{\sqrt{2(\cos u-\cos w)}}. \tag{8.25}$$

Introduce the notation $x = \cos u$ and write the generating function of (8.23) (aside from a constant factor) as

$$\begin{aligned} I(x,t) &:= \int_u^\pi \frac{1}{\sqrt{x-\cos w}} \left[\sum_{n=0}^\infty \sin[(n+\tfrac{1}{2})w]t^n \right] dw \\ &= \int_u^\pi \frac{1}{\sqrt{x-\cos w}} \, \text{Im} \left[e^{iw/2} \sum_{n=0}^\infty (te^{iw})^n \right] dw \\ &= \int_u^\pi \frac{1}{\sqrt{x-\cos w}} \, \text{Im} \left[\frac{e^{iw/2}}{1-te^{iw}} \right] dw \\ &= (1+t) \int_u^\pi \frac{\sin(w/2)\, dw}{\sqrt{x-\cos w}(1-2t\cos w + t^2)} \, dw. \end{aligned}$$

The change of variables $y = \cos w$ gives

$$\begin{aligned} I(x,t) &= \frac{1+t}{\sqrt{2}} \int_{-1}^x \frac{dy}{\sqrt{x-y}\,\sqrt{1+y}(1-2ty+t^2)} \\ &= \frac{1+t}{\sqrt{2}(1+t^2)} \int_{-1}^x \frac{dy}{\sqrt{x-y}\,\sqrt{1+y}\,(1-\beta y)} \end{aligned}$$

with $\beta = 2t/(1+t^2)$. The change of variables $x-y = (1+x)\sin^2\varphi$ and the elementary evaluation

$$\int_0^{\pi/2} \frac{d\varphi}{1+b\sin^2\varphi} = \frac{\pi}{2\sqrt{1+b}} \tag{8.26}$$

confirm that the right-hand side of (8.23) has the same generating function as the Legendre polynomials. This proves the first formula. The same method gives the proof of (8.25).

9. Combinations of Legendre polynomials and trigonometric functions

This section presents two entries in [**12**] that contain the Legendre polynomials in the integrand.

Example 9.1. Entry **7.244.1** states that

$$\int_0^1 P_n(1-2x^2)\sin ax\, dx = \frac{\pi}{2}\left[J_{n+1/2}\left(\frac{a}{2}\right)\right]^2 \tag{9.1}$$

To verify this, let

$$L_n(a) = \int_0^1 P_n(1-2x^2)\sin ax\, dx \tag{9.2}$$

be the left-hand side of (9.1). The recurrence for the Legendre polynomials gives

$$(n+1)P_{n+1}(1-2x^2) = (2n+1)(1-2x^2)P_n(1-2x^2) - nP_{n-1}(1-2x^2). \tag{9.3}$$

Observe that

$$L_n''(a) = -\int_0^1 x^2 P_n(1-2x^2)\sin ax\, dx \tag{9.4}$$

gives

$$L_n(a) + 2L_n''(a) = \int_0^1 (1-2x^2)P_n(1-2x^2)\sin ax\,dx. \tag{9.5}$$

Then (9.3) produces

$$(n+1)L_{n+1}(a) = (2n+1)[L_n(a) + 2L_n''(a)] - nL_{n-1}(a). \tag{9.6}$$

The initial conditions are

$$L_0(a) = \frac{1-\cos a}{a} \text{ and } L_1(a) = \frac{1}{a^3}\left[4 + a^2 + (a^2-4)\cos a - 4a\sin a\right]. \tag{9.7}$$

Now it is shown that

$$R_n(a) = \left[J_{n+1/2}\left(\frac{a}{2}\right)\right]^2 \tag{9.8}$$

the right-hand side of (9.1), without the factor $\pi/2$, satisfies the same recurrence and that the initial values agree with those for $L_n(a)$. The verification is simplified by using the classical recurrence for Bessel functions

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) \tag{9.9}$$

and the relation for derivatives

$$\begin{aligned} 2J_\nu'(x) &= J_{\nu-1}(x) - J_{\nu+1}(x) \\ 4J_\nu''(x) &= J_{\nu-2}(x) - 2J_\nu(x) + J_{\nu+2}(x). \end{aligned} \tag{9.10}$$

The details are unilluminating and they are omitted.

Example 9.2. Entry **7.244.2** states that

$$\int_0^1 P_n(1-2x^2)\cos ax\,dx = \frac{\pi}{2}(-1)^n J_{n+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-n-\frac{1}{2}}\left(\frac{a}{2}\right) \tag{9.11}$$

The proof of this identity can be obtained by the method developed in the previous example. It is convenient to simplify the right-hand side by using the Bessel-Y function

$$Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu} \tag{9.12}$$

to write (9.11) in the form

$$\int_0^1 P_n(1-2x^2)\cos ax\,dx = -\frac{\pi}{2} J_{n+\frac{1}{2}}\left(\frac{a}{2}\right) Y_{n+\frac{1}{2}}\left(\frac{a}{2}\right). \tag{9.13}$$

The details are left to the reader.

10. Combinations of logarithms and trigonometric functions

Section **4.381** contains four entries where the integrands are combinations of $\ln x$ and a basic trigonometric function. The evaluations are expressed in terms of the *cosine integral*

$$\text{ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t}\, dt, \tag{10.1}$$

defined as entry **8.230.2** and the *sine integral*

$$\text{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt \tag{10.2}$$

from entry **8.230.1**. The reader is encouraged to verify the equality of the two expressions in (10.1).

Example 10.1. Entry **4.381.1** is

$$\int_0^1 \ln x \, \sin ax \, dx = -\frac{1}{a}\left(\gamma + \ln a - \text{ci}(a)\right) \tag{10.3}$$

The change of variables $t = ax$ gives

$$\begin{aligned} \int_0^1 \ln x \, \sin ax \, dx &= \frac{1}{a}\int_0^a \ln\left(\frac{t}{a}\right) \sin t \, dt \\ &= \frac{1}{a}\int_0^a \ln t \, \sin t \, dt - \frac{\ln a}{a}\int_0^a \sin t \, dt \\ &= \frac{1}{a}\int_0^a \ln t \, \sin t \, dt - \frac{\ln a}{a}(1 - \cos a). \end{aligned} \tag{10.4}$$

Write the remaining integral as

$$\int_0^a \ln t \, \sin t \, dt = \int_0^a \ln t \, \frac{d}{dt}\left(1 - \cos t\right) \, dt \tag{10.5}$$

and integrate by parts to produce

$$\int_0^a \ln t \, \sin t \, dt = (1 - \cos a) \ln a - \int_0^a \frac{1 - \cos t}{t}\, dt. \tag{10.6}$$

This gives

$$\int_0^1 \ln x \sin ax \, dx = -\frac{1}{a}\int_0^a \frac{1 - \cos t}{t}\, dt \tag{10.7}$$

and the result follows from (10.1). Entry **4.381.3**

$$\int_0^{2\pi} \ln x \, \sin nx \, dx = -\frac{1}{n}\left(\gamma + \ln(2n\pi) - \text{ci}(2n\pi)\right), \tag{10.8}$$

for $n \in \mathbb{N}$, now follows directly from (10.3).

Example 10.2. Entry **4.381.2** is

$$\int_0^1 \ln x \, \cos ax \, dx = -\frac{1}{a}\left(\text{si}(a) + \frac{\pi}{2}\right). \tag{10.9}$$

The proof begins with the change of variables $t = ax$ to produce

$$\begin{aligned} \int_0^1 \ln x \, \cos ax \, dx &= \frac{1}{a}\int_0^a \ln \frac{t}{a} \cos t \, dt \\ &= \frac{1}{a}\int_0^a \ln t \cos t \, dt - \frac{\ln a}{a}\int_0^a \cos t \, dt \\ &= \frac{1}{a}\int_0^a \ln t \cos t \, dt - \frac{\ln a}{a}\sin a. \end{aligned} \tag{10.10}$$

Now integrate by parts to produce

$$\begin{aligned} \int_0^a \ln t \, \cos t \, dt &= \int_0^a \ln t \, \frac{d}{dt}\sin t \, dt \\ &= \ln a \sin a - \int_0^a \frac{\sin t}{t} \, dt \end{aligned}$$

that gives

$$\int_0^1 \ln x \, \cos ax \, dx = -\frac{1}{a}\int_0^a \frac{\sin t}{t} \, dt. \tag{10.11}$$

The result now follows from the definition of the sine integral and the value

$$\int_0^\infty \frac{\sin t}{t} \, dt = \frac{\pi}{2}. \tag{10.12}$$

This is entry **3.721.1** and a variety of proofs appear in [**13**].

The last entry in this section, namely **4.381.4**:

$$\int_0^{2\pi} \ln x \, \cos nx \, dx = -\frac{1}{n}\left(\text{si}(2n\pi) + \frac{\pi}{2}\right) \tag{10.13}$$

follows directly from (10.9).

11. Combinations of Bessel functions and trigonometric functions

There is a variety of entries in [**12**] where the integrand has Bessel and trigonometric functions. Two such entries are presented.

Example 11.1. Entry **6.671.7** states that

$$\int_0^\infty J_0(ax)\sin(bx) \, dx = \begin{cases} 0 & \text{if } 0 < b < a \\ 1/\sqrt{b^2 - a^2} & \text{if } 0 < a < b. \end{cases} \tag{11.1}$$

The proof uses the differential equation for $J_0(x)$:

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \tag{11.2}$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$. The Laplace transform of a function $f(x)$, defined by

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)\, dx \tag{11.3}$$

and this transform satisfies the elementary properties

$$\mathcal{L}[xf(x)] = -\frac{d}{ds} F(s) \tag{11.4}$$

$$\mathcal{L}[f'(x)] = sF(s) - f(0) \text{ and } \mathcal{L}[f''(x)] = s^2F(s) - sf(0) - f'(0). \tag{11.5}$$

Then the Laplace transform of (11.2) then gives

$$\frac{F'(s)}{F(s)} = -\frac{s}{s^2+1}. \tag{11.6}$$

This gives

$$F(s) = \frac{C}{\sqrt{s^2+1}}. \tag{11.7}$$

The value $C = 1$ comes from the standard relation

$$\lim_{x \to 0} f(x) = \lim_{s \to \infty} sF(s), \tag{11.8}$$

applied to $f(x) = J_0(x)$. The value $C = 1$ gives the evaluation

$$\int_0^\infty J_0(x)\, dx = 1. \tag{11.9}$$

Scaling the Laplace transform of $J_0(x)$ gives

$$\int_0^\infty e^{-sx} J_0(ax)\, dx = \frac{1}{\sqrt{s^2+a^2}}. \tag{11.10}$$

Replace s by ib gives

$$\int_0^\infty J_0(ax) \cos bx\, dx - i \int_0^\infty J_0(ax) \sin bx\, dx = \frac{1}{\sqrt{a^2-b^2}}. \tag{11.11}$$

The entry in (11.12) is obtained by matching the imaginary parts. The real parts produce entry **6.671.8**

$$\int_0^\infty J_0(ax) \cos(bx)\, dx = \begin{cases} 1/\sqrt{a^2-b^2} & \text{if } 0 < b < a \\ 0 & \text{if } 0 < a < b. \end{cases} \tag{11.12}$$

12. Combinations of hypergeometric functions and trigonometric functions

This section discusses the evaluation of two entries in [**12**] where the integrand involves trigonometric functions and the hypergeometric function

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!} x^k. \tag{12.1}$$

The reader will find in [**15**] a variety of entries in [**12**] that involve hypergeometric functions. Information about these classical functions can be found in [**3**]. The proofs will involve two well-known transformations:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right) = (1-z)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}\middle| z\right) \tag{12.2}$$

valid for $|\arg(1-z)| < \pi$ and

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-a+b \end{matrix}\middle| \frac{1}{z}\right) + \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}\middle| \frac{1}{z}\right). \end{aligned} \tag{12.3}$$

Example 12.1. Entry **7.531.1** states that

$$\int_0^\infty x \sin \mu x \, {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \frac{3}{2} \end{matrix}\middle| -c^2x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-2} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \tag{12.4}$$

for $\mu > 0$, $\operatorname{Re}\alpha > \frac{1}{2}$, $\operatorname{Re}\beta > \frac{1}{2}$ with its companion entry **7.531.2**

$$\int_0^\infty \cos \mu x \, {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \frac{1}{2} \end{matrix}\middle| -c^2x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-1} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \tag{12.5}$$

for $\mu > 0$, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$.

The proof of (12.4) begins with the observation that the integral is convergent provided $\operatorname{Re}\alpha > 1$ and $\operatorname{Re}\beta > 1$. This follows from (12.3) and the behavior of the hyeprgeometric function at $x = 0$. The integral actually converges for $\operatorname{Re}\alpha > \frac{1}{2}$ and $\operatorname{Re}\beta > \frac{1}{2}$ by using (12.3) to the integrand transformed by (12.2).

In the proof assume first that $\operatorname{Re}\alpha > \frac{3}{2}$ and $\operatorname{Re}\beta > \frac{3}{2}$ and that $c > 0$. The first ingredient in the argument is Parseval's identity

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} F(1-s)G(s), \tag{12.6}$$

where F and G are the Mellin transforms of f and g, respectively. The parameter d is chosen so that the vertical line $\operatorname{Re} s = d$ lies in the intersection of the strips of analyticity of $F(1-s)$ and $G(s)$ and then fix δ so that $d \leqslant \delta < 3$. See [**25**] for details on the Mellin transform. Let

$$f(x) = x \sin \mu x \text{ and } g(x) = {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \frac{3}{2} \end{matrix}\middle| -c^2x^2\right). \tag{12.7}$$

The identity

$$\int_0^\infty x^s \sin \mu x \, dx = \frac{\Gamma(s+1)}{\mu^{s+1}} \cos\left(\frac{\pi s}{2}\right) \tag{12.8}$$

appears, in an equivalent form, as entry **3.761.4**. This gives the Mellin transform of $f(x)$. On the other hand, the evaluation

$$\int_0^\infty t^{s-1} {}_2F_1\left(\left. \begin{matrix} a, b \\ c \end{matrix} \right| -t \right) dt = \frac{\Gamma(c)\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(a)\Gamma(b)\Gamma(c-s)} \tag{12.9}$$

appears in [**3**, p. 86]. Therefore

$$\begin{aligned} \int_0^\infty x^{s-1} {}_2F_1\left(\left. \begin{matrix} \alpha, \beta \\ \frac{3}{2} \end{matrix} \right| -c^2x^2 \right) dx &= \frac{1}{2c^s} \int_0^\infty t^{s/2-1} {}_2F_1\left(\left. \begin{matrix} \alpha, \beta \\ \frac{3}{2} \end{matrix} \right| -t \right) dt \\ &= \frac{\Gamma(3/2)\Gamma(s/2)\Gamma(\alpha - s/2)\Gamma(\beta - s/2)}{2c^s\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\frac{3-s}{2}\right)}, \end{aligned}$$

provided $\min(\operatorname{Re} 2\alpha, \operatorname{Re} 2\beta) > \operatorname{Re} s > 0$.

Parseval's identity now gives

$$\int_0^\infty x \sin \mu x \, {}_2F_1\left(\left. \begin{matrix} \alpha, \beta \\ \frac{3}{2} \end{matrix} \right| -c^2x^2 \right) dx = \\ \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(2-s)\sin(\pi s/2)}{\mu^{2-s}} \frac{\Gamma(3/2)\Gamma(s/2)\Gamma(\alpha - s/2)\Gamma(\beta - s/2)}{2c^s\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\frac{3-s}{2}\right)} ds$$

Now insert the factor $\Gamma(1-s/2)$ both in the numerator and denominator of the previous integrand and use the reflection formula for the gamma function in the form

$$\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(1 - \tfrac{s}{2}\right) = \frac{\pi}{\sin(\pi s/2)} \tag{12.10}$$

and the duplication formula for the gamma function

$$\Gamma\left(1 - \tfrac{s}{2}\right)\Gamma\left(\tfrac{3-s}{2}\right) = \frac{\sqrt{\pi}}{2^{1-s}}\Gamma(2-s) \tag{12.11}$$

to obtain

$$\int_0^\infty x \sin \mu x \, {}_2F_1\left(\left. \begin{matrix} \alpha, \beta \\ \frac{3}{2} \end{matrix} \right| -c^2x^2 \right) dx = \\ \frac{\sqrt{\pi}\,\Gamma(3/2)}{2\mu^2\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma\left(\alpha - \tfrac{s}{2}\right)\Gamma\left(\beta - \tfrac{s}{2}\right) 2^{1-s} \left(\frac{\mu}{c}\right)^s ds, \tag{12.12}$$

valid for $1 < \operatorname{Re} s \leqslant d < \delta \leqslant 3$. The next step is to use the evaluation

$$\frac{1}{2\pi i} \int_{m-i\infty}^{m+i\infty} 2^{s-2} a^{-s} \Gamma\left(\frac{s}{2} + \frac{\nu}{2}\right) \Gamma\left(\frac{s}{2} - \frac{\nu}{2}\right) x^{-s} \, ds = K_\nu(ax) \tag{12.13}$$

that may be found in [**23**, p. 115]. To use this entry, replace s by $\alpha + \beta - s$ in (12.12) to produce the result.

The proof of (12.5) is similar, so the details are left to the reader.

Acknowledgments. The sixth author acknowledges the partial support of NSF-DMS 0713836. The second author is a postdoctoral fellow partially supported by the same grant. The third and fourth authors are graduate students partially supported by this grant. The work of the fifth author is partially supported by the Natural Sciences and Engineering Research Council of Canada. The work of the last author was partially supported by the iCODE Institute, Research Project of the Idex Paris-Saclay. The authors wish to thank D. Zeilberger for the automatic proof presented in Note 5.5.

References

[1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Dover, New York, 1972.
[2] T. Amdeberhan, V. Moll, J. Rosenberg, A. Straub, and P. Whitworth. The integrals in Gradshteyn and Ryzhik. Part 9: Combinations of logarithmic, rational and trigonometric functions. *Scientia*, 17:27–44, 2009.
[3] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 1999.
[4] W. N. Bailey. *Generalized hypergeometric series*. Cambridge University Press, 1935.
[5] I. V. Blagouchine. Rediscovering of Malmsten's integrals, their evaluation by contour integration methods and some related results. *The Ramanujan Journal*, 35:21–110, 2014.
[6] K. Boyadzhiev, L. Medina, and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 11: The incomplete beta function. *Scientia*, 18:61–75, 2009.
[7] W. Chu. Analytical formulae for extended ${}_3F_2$-series of Watson-Whipple-Dixon with two extra integer parameters. *Math. Comp.*, 81:467–479, 2012.
[8] M. W. Coffey. Functional equations for the Stieltjes constants. *The Ramanujan Journal*, To appear, 2015.
[9] A. Dixit, L. Jiu, V. Moll, and C. Vignat. The finite Fourier transform of classical polynomials. *Journal of the Australian Math. Soc.*, 98:145–160, 2015.
[10] I. Gessel and D. Stanton. Strange evaluations of hypergeometric series. *SIAM J. Math. Anal.*, 24:24–33, 1978.
[11] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.
[12] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by D. Zwillinger and V. Moll. Academic Press, New York, 8th edition, 2015.
[13] G. H. Hardy. The integral $\int_0^\infty \frac{\sin x}{x}\, dx$. *Math. Gazette*, 5:98–103, 1909.
[14] Y. Katznelson. *An introduction to Harmonic Analysis*. Cambridge University Press, New York, 2004.
[15] K. Kohl and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 20: Hypergeometric functions. *Scientia*, 21:43–54, 2011.
[16] Y. L. Luke. *The special functions and their approximations*, volume 1. Academic Press, New York-London, 1969.
[17] L. Medina and V. Moll. A class of logarithmic integrals. *The Ramanujan Journal*, 20:91–126, 2009.
[18] L. Medina and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 10: The digamma function. *Scientia*, 17:45–66, 2009.
[19] M. Milgram. 447 instances of hypergeometric ${}_3F_2(1)$. *ArXiv: abs/1105.312*, 2015.
[20] B. Minton. Generalized hypergeometric functions of unit argument. *J. Math. Phys.*, 11:1375–1376, 1970.
[21] V. Moll. The integrals in Gradshteyn and Ryzhik. Part 1: A family of logarithmic integrals. *Scientia*, 14:1–6, 2007.
[22] I. Nemes, M. Petkovsek, H. Wilf, and D. Zeilberger. How to do MONTHLY problems with your computer. *Amer. Math. Monthly*, 104:505–519, 1997.

[23] F. Oberhettinger. *Tables of Mellin transforms*. Springer-Verlag, New York, 1974.
[24] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.
[25] R. B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes Integrals*, volume 85 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 2001.
[26] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series, volume 3: More Special Functions*. Gordon and Breach Science Publishers, 1990.
[27] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series. Five volumes.* Gordon and Breach Science Publishers, 1992.
[28] C. Schneider. Symbolic summation assists Combinatorics. *Séminaire Lotharingien de Combinatoire*, 56:1–36, April 2007. Article B56b.
[29] I. Vardi. Integrals, an Introduction to Analytic Number Theory. *Amer. Math. Monthly*, 95:308–315, 1988.
[30] G. N. Watson. A note on generalized hypergeometric series. *Proc. London Math. Soc.*, 23:104–114, 1925.
[31] J. Wimp. Irreducible recurrences and representation theorems for ${}_3F_2(1)$. *Comp. Math. Appl.*, 9:669–678, 1983.

Department of Mathematics,Tulane University, New Orleans, LA 70118
E-mail address: tamdeber@tulane.edu

Department of Mathematics,Tulane University, New Orleans, LA 70118
E-mail address: adixit@tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: xguan1@math.tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: ljiu@math.tulane.edu

Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada, M3J 1P3
E-mail address: kuznetsov@mathstat.yorku.ca

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118 and, Dept. of Physics, Universite Orsay Paris Sud, L. S. S./Supelec, France
E-mail address: cvignat@tulane.edu

Received ??, revised ??

Departamento de Matemática
Universidad Técnica Federico Santa María
Casilla 110-V,
Valparaíso, Chile