Problem #12305. Proposed by S. Sharma (India). Let $\gamma$ be the Euler-Mascheroni constant. Prove
\[
\int_0^1 \frac{x - 1 - x \log x}{x \log x - x \log^2 x} \, dx = \gamma.
\]

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA.

First off, we have $\gamma = \int_0^1 \left( \frac{1}{\log x} + \frac{1}{1-x} \right) \, dx$ as one of the integral representations for $\gamma$. With this in mind, it suffices to show the difference between the two integral is zero. That means, evaluate
\[
\int_0^1 \left( \frac{x - 1 - x \log x}{x \log x - x \log^2 x} - \frac{1}{\log x} + \frac{1}{1-x} \right) dx = -\int_0^1 \left( \frac{1}{x \log x - x \log^2 x} + \frac{1}{1-x} \right) dx.
\]

So, we focus on last integral. This, however, goes as follows: since
\[
\int \left( \frac{1}{x \log x - x \log^2 x} + \frac{1}{1-x} \right) dx = \log \left( \frac{\log x}{(1-x)(\log x - 1)} \right),
\]
we compute two limits: $x \to 0^+$ and $x \to 1^-$. Both are executed via L'Hôpital's Rule, resulting in
\[
\lim_{x \to 0^+} \log \left( \frac{\log x}{(1-x)(\log x - 1)} \right) = 0 \quad \text{and} \quad \lim_{x \to 1^-} \log \left( \frac{\log x}{(1-x)(\log x - 1)} \right) = 0.
\]
The proof is complete. $\square$