Problem #12304. Proposed by M. Bataille (France). Let $m$ and $n$ be positive integers with $m < n$. Prove

$$\left(\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{n-k}\right) \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1}\right) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(n-k)(k+1)}.$$ 

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Replacing $n$ by an indeterminate $x$, we intend to justify the equality between two rational functions (meromorphic functions with simple poles at $x = 0, 1, \ldots, m$). That is to say,

$$\left(\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{x-k}\right) \left(\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{k+1}\right) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(x-k)(k+1)}.$$ 

Clearly, these simple poles are shared by both sides. Hence, it suffices to compare the coefficients for $\frac{1}{x^j}$ for $j = 0, 1, \ldots, m$. Fix such $j$. The claim, then, amounts to

$$\binom{m}{j} (-1)^j \sum_{k=0}^{m} \binom{j}{k} \frac{(-1)^k}{k+1} = \binom{m}{j} (-1)^j \quad \iff \quad \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k}{k+1} = \frac{1}{j+1}.$$ 

Starting with $\sum_{k=0}^{j} \binom{j}{k} (-1)^k x^k = (1 - x)^j$, integrate both sides over the interval $0 \leq x \leq 1$ to get

$$\sum_{k=0}^{j} \binom{j}{k} (-1)^k \int_{0}^{1} x^k \, dx = \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k}{k+1} = \int_{0}^{1} (1 - x)^j \, dx = \frac{1}{j+1}.$$ 

The proof is complete. \square

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