Problem #12293. Proposed by H. Ohtsuka (Japan) and R. Tauraso (Italy). For any integer \( n \geq 1 \), and any real number \( r > 0 \), prove

\[
\sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{k} r^j \binom{n}{j} \right) \left( \sum_{j=0}^{k} (-r)^j \binom{n}{j} \right) = \left( \frac{(r+1)^n + (r-1)^n}{2} \right)^2 .
\]

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA.

From Binomial Theorem, \( \frac{(r+1)^n + (r-1)^n}{2} = \sum_{j=0}^{[n/2]} r^n 2j \binom{n}{2j} \). Using the identity \( ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2 \) with \( a = \sum_{j=0}^{k} r^j \binom{n}{j} \) and \( b = \sum_{j=0}^{k} (-r)^j \binom{n}{j} \), we proceed as:

\[
\sum_{k=0}^{n} (-1)^k a \cdot b = \sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{k} r^j \binom{n}{j} + (-r)^j \binom{n}{j} \right)^2 - \sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{k} r^j \binom{n}{j} - (-r)^j \binom{n}{j} \right)^2
\]

\[
= \sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{[k/2]} r^{2j} \binom{n}{2j} \right)^2 - \sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{[(k-1)/2]} r^{2j+1} \binom{n}{2j+1} \right)^2 .
\]

Successive terms in \( \sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{[k/2]} \right)^2 \) as well as in \( \sum_{k=1}^{n} (-1)^k \left( \sum_{j=0}^{[(k-1)/2]} \right)^2 \) cancel pair-wise. Assume \( n \to 2n \) is even. In this case, second the double sum vanishes while the first double sum retains one summand (for \( k = 2n \)), i.e. \( \left( \sum_{j=0}^{n} r^{2j} \binom{2n}{2j} \right)^2 \). This agrees with \( \left( \frac{(r+1)^{2n} + (r-1)^{2n}}{2} \right)^2 = \left( \sum_{j=0}^{n} r^{2n-2j} \binom{2n}{2j} \right)^2 \). The argument is similar if \( n \to 2n + 1 \) is odd. This time the first double sum vanishes while the second double sum maintains a single summand (for \( k = 2n + 1 \)), i.e. \( \left( \sum_{j=0}^{n} r^{2j+1} \binom{2n+1}{2j+1} \right)^2 \). Again, this matches \( \left( \frac{(r+1)^{2n+1} + (r-1)^{2n+1}}{2} \right)^2 = \left( \sum_{j=0}^{n} r^{2n+1-2j} \binom{2n+1}{2j} \right)^2 \).

The proof is now complete. \( \square \)