Problem #12286. Proposed by I. Gessel (USA). Let \( p \) be a prime number, and let \( m \) be a positive integer not divisible by \( p \). Show that the coefficients of \((1+x+\cdots+x^{m-1})^{p-1}\) that are not divisible by \( p \) are alternately 1 and \(-1\) modulo \( p \). For example, \((1+x+x^2+x^3)^4 \equiv 1-x+x^4-x^6+x^8-x^{11}+x^{12} \mod 5\).

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. The case \( p = 2 \) is trivial, so assume \( p \) is odd. Notice \( 1+x^m+\cdots+x^{(p-1)m} \equiv (1-x^m)^{p-1} \mod p \). Then, modulo \( p \), we have

\[
(1 + x + \cdots + x^{m-1})^{p-1} = \left(1 - \frac{x^m}{1-x}\right)^{p-1} = (1-x) \frac{(1-x^m)^{p-1}}{(1-x)^p} \equiv (1-x) \frac{1+x^m+\cdots+x^{(p-1)m}}{1-x^p} \equiv (1-x) \frac{1+x^m+\cdots+x^{(p-1)m}}{1-x^p}
\]

Observe that each term \( x^{mk+pj} \) appears only once since \( \gcd(m,p) = 1 \) and \( 0 \leq k < p \). Thus, the non-vanishing summands in the above sum \( \sum_{k,j} x^{mk+pj} \) all have coefficients equal to 1. Upon multiplying by \((1-x)\), the assertion follows immediately. \( \square \)