Solution to Problem #12276

Problem #12276. Proposed by J. Santmyer, Las Cruces, NM. Prove

\[
\sum_{n=2}^{\infty} \frac{1}{n+1} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{2^{i-1}(i-1)!((n-2i))!} = 1.
\]

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA.

Start with the exponential generating function for the number of involutions \( I_m \), given in the form

\[
e^x + \frac{1}{2} x^2 = \sum_{m=0}^{\infty} \frac{I_m}{m!} x^m.
\]

It is also well-known \( I_m = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m!}{2^{k!}((m-2k))!} \). Now, proceed as follows:

\[
x^2 e^{x + \frac{1}{2} x^2} = \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{x^{m+2}}{2^k!(m-2k)!} = \sum_{n=2}^{\infty} \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \frac{x^n}{2^k!(n-2-2k)!}
\]

Integrate both sides over the range \( 0 \leq x \leq 1 \) to find

\[
\int_0^1 x^2 e^{x + \frac{1}{2} x^2} \, dx = (x - 1)e^{x + \frac{1}{2} x^2} \bigg|_0^1 = 1
\]

while

\[
\int_0^1 \sum_{n=2}^{\infty} x^n \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{2^{i-1}(i-1)!((n-2i))!} \, dx = \sum_{n=2}^{\infty} \frac{1}{n+1} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{2^{i-1}(i-1)!((n-2i))!}
\]

The proof is complete. □