Problem #12106. Proposed by H. Ohtsuka, Japan. For any positive integer \( n \), prove

\[
\sum_{k=1}^{n} \binom{n}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = \sum_{1 \leq i \leq j \leq n} \frac{2^n - 2^{n-i}}{ij}.
\]

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Denote the LHS and RHS by \( f_n \) and \( g_n \), respectively, and induct on \( n \). It’s obvious \( f_1 = g_1 \). Assume \( f_n = g_n \). For the case \( n + 1 \), proceed with \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \) and split the sum

\[
f_{n+1} = \sum_{k=1}^{n+1} \binom{n}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} + \sum_{k=1}^{n+1} \binom{n}{k-1} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = f_n + 1 + \sum_{k=1}^{n} \binom{n}{k} \sum_{1 \leq i \leq j \leq k+1} \frac{1}{ij}
\]

\[
= 2f_n + 1 + \sum_{k=0}^{n} \binom{n}{k+1} \sum_{i=1}^{k+1} \frac{1}{i} = 2f_n + 1 + \sum_{k=0}^{n} \binom{n}{k+1} \sum_{i=1}^{k+1} \frac{1}{i}
\]

\[
= 2f_n + \sum_{k=0}^{n} \frac{1}{k+1} \int_{0}^{1} \frac{1-x^k}{1-x} \frac{dx}{1-x} = 2f_n + n + 1 \int_{0}^{1} \frac{1-x^{n+1}}{1-x} \frac{dx}{1-x} \sum_{k=0}^{n} \frac{n+1}{k+1} (1-x^{k+1})
\]

\[
= 2f_n + \frac{1}{n+1} \int_{0}^{1} \frac{2^{n+1} - (1+x)^{n+1}}{1-x} \frac{dx}{1-x} = 2f_n + \frac{1}{n+1} \int_{0}^{1} \sum_{i=1}^{n+1} \frac{2^{n+1-i} (1+x)^{i-1}}{i} dx
\]

\[
= 2f_n + \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{2^{n+1-i} - 2^{n+1-i}}{i}
\]

On the other hand, it is easy to see \( g_{n+1} = 2g_n + \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{2^{n+1-i} - 2^{n+1-i}}{i} \). The proof follows. □