High Order Interpolation Kernels With Explicit Formulas

Robert Blake
rblake@tulane.edu
Biomedical Engineering Department
Tulane University
New Orleans, LA 70118

Ricardo Cortez
rcortez@tulane.edu
Mathematics Department
Tulane University
New Orleans, LA 70118

Christopher Duncan
duncan@math.tulane.edu
Mathematics Department
Tulane University
New Orleans, LA 70118

Robert Miller
Mathematics Department
University of Louisiana at Lafayette
P.O. Box 41010
Lafayette, LA 70504

Luis O. Pérez
Mathematics Department
University of Puerto Rico at Humacao
Humacao, PR 00791

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Abstract

The problem of reconstructing a smooth function or one of its derivatives from the function values at the nodes of a uniform grid is addressed. The approximations are given by a linear combination of the grid values of the function, where the weights come from a piecewise polynomial kernel. Two classes of compactly supported kernels are considered. The first one, smooth kernels, consists of piecewise polynomials of degree ℓ whose derivatives up to $\ell - 1$ are continuous. Our results show that for a given polynomial degree, there is a maximum accuracy that can be achieved as long as the support of the kernel is larger than a minimum value. If the support is equal to this minimum value, then the kernel is unique. The second class, narrow kernels, consists of kernels whose regularity is not enforced and all degrees of freedom are used to provide as high an accuracy as possible in the approximation. In this case, given the desired support of the kernel, there is a maximum accuracy that can be achieved as long as the polynomial degree is larger than a minimum value. If the polynomial degree equals this minimum value, the kernel is unique. We also show that although no regularity is enforced for these kernels, they are continuous and even. Similar results are proven for kernels used to approximate $f'(x)$ given the values of $f(x)$ on the uniform grid. Not all of the latter kernels are derivatives of interpolation kernels and some are discontinuous. Numerical examples in one and two dimensions are provided to illustrate the performance of the kernels.

1 Introduction

A common computational problem with practical applications is to establish the intermediate properties of a function from a given discrete set of data. A typical situation is to use function values at grid points to determine information about the function or its derivatives at intermediate points. Some well known interpolation methods include polynomial approximations and splines, which are used to approximate the function and sometimes its derivatives. Another approach is to use a set of basis kernels and represent the function as a linear combination of the kernels centered at different points. This technique is at the core of finite element methods and is also used in particle methods for solving PDEs [13, 6, 25, 28, 29] as well as in other contexts such as Lagrangian vortex methods for fluid flow [2, 16, 22, 14, 10], particle strength exchange methods [12, 7, 9], smooth particle hydrodynamics [20, 21], the diffusion velocity method [18], the impulse method [3, 5, 8], the immersed boundary method [23, 27] and in fast summation methods like the method of local corrections [1]. These methods seek to update variable values at arbitrary positions by solving a set of PDEs on a grid, so there is a need to transfer information from the particle positions to the grid and back. Overset grid methods face a similar challenge of communicating information from one grid to another.

We focus our attention on the problem of approximating the value of a function or one of its derivatives at an arbitrary point from the function values on a uniform grid of size h. Our approach is to design a kernel $\Lambda_h(x)$ so that the s-th derivative of the function can be approximated by

$$f^{(s)}(x) \approx \frac{1}{h^s} \sum_{k=-\infty}^{\infty} f_k \; \Lambda_h(x - y_k) h$$

where f_k are the function values at the grid points y_k. The kernel $\Lambda_h(x)$ is assumed to be piecewise polynomial with compact support. One can interpret this formula as a quadrature rule where the kernel provides the weights for the approximation. Alternatively, one can think of the kernel as a smooth approximation of a distribution and the sum as the discretization of a convolution. We concentrate on $s = 0$ most of the time because the results extend to the case $s > 0$ with some modifications. This is explained in Section 6.

The principal issue we address is to determine how one can design kernels $\Lambda_h(x)$ with a prescribed degree of regularity and in such a way that the approximation above have a given order of accuracy. For accuracy, the kernel must satisfy discrete moment conditions [4]. Our results show that regularity and accuracy are coupled so that after imposing regularity, the kernel already meets some of the accuracy conditions. Conversely, it has been observed that imposing accuracy conditions yields a kernel that is continuous [24].

There is important work on piecewise polynomial kernels of the type considered here. In particular, the Z-spline $Z_m(x)$ is a $C^{m-1}(\mathbb{R})$ function made of polynomials of degree $\ell = 2m - 1$. It is known that $Z_m(x)$ is an interpolation kernel with order of accuracy $2m-1$ and support $[-m, m]$. The order of accuracy and smallness of the support are optimal [26]. In other words, there is a unique spline of degree $\ell = 2m - 1$ in $C^{m-1}(\mathbb{R})$ with the minimum compact support $[-m, m]$ which approximates a function $f(x)$ with error $O(h^{2m-1})$.

We concentrate on either even or odd kernels consisting of piecewise polynomials of an arbitrary degree on a uniform grid. This choice in turn avoids the global complications faced when considering the general polynomial approximation and the Runge example [11]. We investigate interpolation kernels in two extreme cases: (1) when the maximum amount of regularity is enforced first and any remaining degrees of freedom are used to enforce as many moment conditions as possible; and (2) when no regularity is enforced, only moment conditions. In the latter case, the kernels may be discontinuous at the grid points. Both cases are important. In a typical interpolation problem, where the kernel simply provides the weights for the approximation, the regularity of Λ_h is not a concern and one can achieve a given accuracy with a more compact kernel than if regularity is required. On the other hand, the interpolation is often one step within a larger problem for which a minimum regularity of the interpolation may be necessary.

In the current work, we discuss the tradeoff between accuracy and regularity for a given support size and polynomial degree. We establish the moment preserving properties of the kernels and the accuracy they achieve. We also perform numerical tests to verify the analytical results and work a few examples to illustrate the main points.

2 Preliminaries

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$. Given $h > 0$ let y_k be the points on a uniform grid $y_k = kh$. Define the function $(L_s f) : \mathbb{R} \to \mathbb{R}$ for $s = 0, 1, \dots,$ as*

$$(L_s f)(x) = \frac{1}{h^s} \sum_{k=-\infty}^{\infty} f(y_k) \Lambda_h(x - y_k) h,$$

where Λ is the kernel for L_s and Λ_h is defined by $\Lambda_h(x) = \dfrac{1}{h} \Lambda(x/h)$.

Definition 2. *For $h > 0$ and $y_k = kh$, the p^{th} discrete moment of $\Lambda_h(x)$ is the function given by*

$$M_p(\Lambda_h; x) = \sum_{k=-\infty}^{\infty} (x - y_k)^p \Lambda_h(x - y_k) h.$$

We mention that since Λ has compact support, the sum over k only has a finite number of nonzero terms at the grid points near x. For this reason we can think of $x \in [0, h)$ without loss of generality. It is easy to see by making the substitution $x = zh$ with $0 \le z < 1$, that $M_p(\Lambda_h; x) = h^p M_p(\Lambda; z)$.

The function $(L_s f)(x)$ can be written in terms of the moments of Λ by assuming $f \in C^n(\mathbb{R})$ and substituting the Taylor series

$$f(y_k) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x)}{j!} (y_k - x)^j + \frac{f^{(n)}(c_k)}{n!} (y_k - x)^n$$

into the definition of $(L_s f)(x)$,

$$\begin{aligned}
(L_s f)(x) &= \frac{1}{h^s} \sum_{j=0}^{n-1} \frac{f^{(j)}(x)}{j!} \sum_{k=-\infty}^{\infty} (y_k - x)^j \Lambda_h(x - y_k) h + \frac{1}{h^s} \sum_{k=-\infty}^{\infty} \frac{f^{(n)}(c_k)}{n!} (y_k - x)^n \Lambda_h(x - y_k) h \\
&= \frac{1}{h^s} \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} f^{(j)}(x) M_j(\Lambda_h; x) + O(h^{n-s}) \\
&= \frac{1}{h^s} \sum_{j=0}^{n-1} \frac{(-h)^j}{j!} f^{(j)}(x) M_j(\Lambda; x) + O(h^{n-s}).
\end{aligned}$$

The last equation shows that in order for $(L_s f)(x)$ to approximate $f^{(s)}(x)$, the kernel must satisfy

$$M_j(\Lambda; x) = (-1)^s \; s! \; \delta_{j,s} \quad \text{for } j = 0, 1, \ldots, s, \ldots, m \leq n-1$$

and the approximation will have an error $O(h^{m+1-s})$. Here $\delta_{j,s}$ is the Kronecker delta. For example, to interpolate f between grid points (i.e. $s = 0$), one should choose Λ so that $M_j(\Lambda; x) = \delta_{j,0}$ for $j = 0, 1, \cdots, m \leq n-1$. Then

$$(L_0 f)(x) = f(x) + O(h^{m+1}).$$

Similarly, to approximate $f'(x)$ with $(L_1 f)(x)$, one should choose Λ such that $M_j(\Lambda; x) = -\delta_{j,1}$ for $j = 0, 1, 2, \cdots, m \leq n-1$ to get

$$(L_1 f)(x) = f'(x) + O(h^m).$$

In this way the kernel Λ determines the properties of the operator. Throughout this paper, we will refer to Λ as an interpolation kernel in the specific case $s = 0$, otherwise we will call it an approximation kernel.

The Particle Strength Exchange (PSE) method [12, 7, 9] was developed for advection-diffusion equations so that the kernel (in one dimension) is designed to approximate second derivatives, satisfying $M_j(\Lambda; x) = 2\delta_{j,2}$ for $j = 0, 1, \cdots, m \leq n-1$. The only difference is that the PSE kernels satisfy continuous moment conditions written in terms of integrals rather than discrete moments.

We conclude this section with a counting argument for an upper bound on the number of conditions required for an interpolation kernel of order $m+1$. We assume that the kernel Λ_h is piecewise polynomial of degree ℓ. From the previous arguments, this requires $M_p(\Lambda_h; x) = \delta_{p,0}$ for $0 \leq p \leq m$. By Definition 2, $M_p(\Lambda_h; x)$ is a polynomial of degree $(p+\ell)$, and thus its $(p+\ell+1)$ coefficients must be set to the appropriate values. To do this for $0 \leq p \leq m$ would require $(\ell+1)(m+1) + \frac{1}{2}m(m+1)$ conditions.

However, this number overestimates the number of independent constraints. We can see this by considering

$$N_p(\Lambda; z) = \sum_{k \in \mathbb{Z}} k^p \Lambda(z-k)$$

which is motivated by the fact that we would like $(L_0 f)(x) = f(x)$ when $f(x) = x^p$ and $0 \leq p \leq m$. The reason for this is that any smooth function can be expanded in a Taylor series and a kernel of order $m+1$ would have to be exact for all polynomials of degree m or less. The following proposition shows that there is a one-to-one correspondence between $N_p(\Lambda; z)$ and $M_p(\Lambda; z)$. In other words, requiring $M_p(\Lambda; z) = \delta_{p,0}$ is equivalent to requiring $N_p(\Lambda; z) = z^p$, but $N_p(\Lambda; z)$ is a polynomial of degree ℓ so that a piecewise polynomial kernel of order $m+1$ requires at most $(\ell+1)(m+1)$ conditions.

Proposition 1. *Let $p \geq 0$ and $0 \leq z < 1$. Define*

$$N_0(\Lambda; z) = \sum_{k \in \mathbb{Z}} \Lambda(z-k) \qquad \textit{and} \qquad N_p(\Lambda; z) = \sum_{k \in \mathbb{Z}} k^p \Lambda(z-k) \; \textit{for } p \geq 1.$$

Then $M_p(\Lambda; z)$ and $N_p(\Lambda; z)$ satisfy the relations,

$$\begin{aligned} M_p(\Lambda; z) &= \sum_{j=0}^{p} \binom{p}{j} (-1)^j N_j(\Lambda; z) z^{p-j} \\ N_p(\Lambda; z) &= \sum_{j=0}^{p} \binom{p}{j} (-1)^j M_j(\Lambda; z) z^{p-j}. \end{aligned}$$

Proof. To simplify the notation in the proof we write M_p and N_p for $M_p(\Lambda; x)$ and $N_p(\Lambda; x)$. Clearly, $M_0 = N_0$. The first identity follows by expanding $(z-k)^p$ in the definition of M_p. The second identity follows from the first.

$$\begin{aligned}\sum_{j=0}^{p}(-1)^j\binom{p}{j}z^{p-j}M_j &= \sum_{j=0}^{p}\sum_{i=0}^{j}\binom{p}{j}\binom{j}{i}(-1)^{j+i}z^{p-i}N_i \\ &= \sum_{i=0}^{p}N_i z^{p-i}\sum_{j=i}^{p}\binom{p}{j}\binom{j}{i}(-1)^{j+i}.\end{aligned}$$

Since $\binom{p}{j}\binom{j}{i} = \binom{p-i}{j-i}\binom{p}{i}$, we have

$$\begin{aligned}\sum_{j=0}^{p}\binom{p}{j}(-1)^j z^{p-j}M_j &= \sum_{i=0}^{p}\binom{p}{i}N_i z^{p-i}\sum_{j=i}^{p}\binom{p-i}{j-i}(-1)^{j-i} \\ &= \sum_{i=0}^{p}\binom{p}{i}N_q z^{p-i}\sum_{j=0}^{p-i}\binom{p-i}{j}(-1)^j \\ &= \sum_{i=0}^{p}\binom{p}{i}N_q z^{p-i}\delta_{p,i} \\ &= N_p.\end{aligned}$$

□

Based on this proposition and the preceding arguments we conclude that $(L_s f)(x)$ approximates $f^{(s)}(x)$ with order of accuracy $O(h^m)$ if and only if

$$N_p(\Lambda; x) = \begin{cases} 0 & \text{for} \quad 0 \le p < s \\ \frac{p!}{(p-s)!}\, z^{p-s} & \text{for} \quad s \le p < m+s. \end{cases} \tag{1}$$

3 Piecewise Polynomial Kernels

We now turn our attention to a class of compactly supported piecewise polynomial kernels that satisfy the moment conditions mentioned above. First we introduce some notation. Let S_R^ℓ be the set of all functions $\Lambda : \mathbb{R} \to \mathbb{R}$ such that $supp(\Lambda) = [-R, R]$ and Λ is piecewise polynomial of degree ℓ with nodes at the integers. That is, for $k = -R, 1-R, \ldots, R-1$ and $x \in [k, k+1)$

$$\Lambda(x) = P_k(x) = \sum_{j=0}^{\ell} a_{k,j}(x-k)^j. \tag{2}$$

Here, ℓ and R can be taken as independent parameters although one is usually interested in the smallest possible value of R for a given ℓ. A third parameter is the regularity r of the kernel. For instance, we are interested in a kernel $\Lambda \in S_R^\ell \cap C^r(\mathbb{R})$ and we ask: *What is the highest order of the approximation that can be achieved with a given set of parameters?*

In [26], Becerra Sagredo discusses kernels called Z-splines. The Z-spline $Z_R(x)$ is the unique kernel in $S_R^{2R-1} \cap C^{R-1}(\mathbb{R})$ which produces an $O(h^{2R-1})$ interpolation. These functions have a certain balance between the degree of regularity and the order of accuracy. In the next subsections we consider two extreme cases. First, given ℓ, we consider piecewise polynomial kernels of degree ℓ that are as smooth as possible (i.e. $\Lambda \in C^{\ell-1}(\mathbb{R})$). We show that the maximum order of accuracy is $O(h^{\ell+1})$ and the support of the kernel must satisfy $R \geq 2\lfloor \ell/2 \rfloor + 1$. For a given degree ℓ, these functions are nearly twice as smooth as the Z-splines and slightly more accurate, but their support is almost twice as large.

Next, we take up questions at the other extreme: *Given a radius of support R, what is the maximum order of accuracy that can be achieved? What is the smallest degree of the resulting kernels? How smooth are they?* We show that even though no continuity conditions are imposed, the kernel (with smallest degree) is in $S_R^{2R-1} \cap C^0(\mathbb{R})$, and provides an $O(h^{2R})$ interpolation. These kernels have very similar properties as the Z-splines although the extra order of accuracy comes at the expense of $R-1$ degrees of regularity.

3.1 Smooth Kernels

We consider those even kernels $\Lambda \in S_R^\ell$ which are as smooth as possible. If we require more than $\ell-1$ degrees of regularity, then the polynomial at the edge of the support of Λ must be zero in order for it to smoothly join zero at the edge of the support. The same argument shows that the next polynomial piece must also be identically zero, and so on. Thus, given ℓ, we require $\Lambda \in C^{\ell-1}(\mathbb{R})$.

The polynomial coefficients will be determined by imposing regularity, symmetry, and moment conditions. Since Λ is assumed to be an even function, there are only R different polynomials of degree ℓ that must be determined. This results in $R(\ell+1)$ coefficients. Requiring $\Lambda \in C^{\ell-1}(\mathbb{R})$ represents ℓ conditions at the nodes $x = 1, 2, \ldots, R$ for a total of $R\ell$ conditions. Additionally, since $\Lambda(x)$ is even and $C^{\ell-1}(\mathbb{R})$, we conclude that the coefficients of all odd powers of x in $P_0(x)$ which are less than ℓ must be zero. There are $\lfloor \ell/2 \rfloor$ of them. Thus, regularity and symmetry account for a total of $R\ell + \lfloor \ell/2 \rfloor$ conditions, leaving $R - \lfloor \ell/2 \rfloor$ degrees of freedom that are used to satisfy moment conditions. This argument also shows that there is a minimum support necessary for high-order approximations.

The next step is to investigate how many conditions are required to satisfy $M_0(\Lambda; x) = 1$ and $M_p(\Lambda; x) = 0$ for as many $p > 0$ as possible. We start with an example.

Consider the well-known hat function

$$\Lambda_1(x) = \begin{cases} 1 - |x| & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$$

which belongs to $S_1^1 \cap C^0(\mathbb{R})$. In this case the requirement $M_0(\Lambda_1; x) = 1$ is a single condition which, together with the continuity condition at $x = 1$, is enough to determine the two coefficients. Surprisingly, it turns out that for this function $M_1(\Lambda_1; x) = 0$ automatically while $M_2(\Lambda_1; x) = x(1-x) \neq 0$ so that this kernel provides a second order interpolation formula. It is natural to consider a piecewise linear kernel with $R > 1$ so that the additional coefficients may provide the degrees of freedom to satisfy higher discrete moments. However, this is not possible because one discovers that $M_2(\Lambda_1; x) = x(1-x) + C_R$ regardless of R.

We investigated empirically kernels in $S_R^\ell \cap C^{\ell-1}(\mathbb{R})$ for $\ell = 1, \ldots, 8$ and summarized the findings in Table 1. The row labeled M_0 gives the number of conditions needed to enforce $M_0(\Lambda; x) = 1$. Notice that it takes only one condition to do it regardless of the value of ℓ. The other rows show the additional number of conditions required to set those moments to zero once $M_0(\Lambda; x)$ has been set to 1. The odd moments are automatically zero from the previous requirements so they represent no additional conditions. A line in a cell indicates that the corresponding moment cannot be set to zero regardless of the choice of R because it contains nontrivial polynomial terms. For the cases shown in the table, one can enforce $M_p(\Lambda; x) = \delta_{p,0}$ for $0 \le p \le \ell$ as long as R is large enough. The minimum value of R (a smallest support of Λ) for which $\Lambda \in S_R^\ell \cap C^{\ell-1}(\mathbb{R})$ results in an interpolant of order $\ell + 1$ is $R = 2\lfloor \ell/2 \rfloor + 1$.

Smooth Even Kernels

	ℓ							
	1	2	3	4	5	6	7	8
M_0	1	1	1	1	1	1	1	1
M_1	0	0	0	0	0	0	0	0
M_2	—	1	1	1	1	1	1	1
M_3		—	0	0	0	0	0	0
M_4			—	1	1	1	1	1
M_5				—	0	0	0	0
M_6					—	1	1	1
M_7						—	0	0
M_8							—	1
M_9								—

Table 1: The row labeled M_0 gives the number of conditions needed to enforce $M_0(\Lambda; x) = 1$. The other rows show the additional number of conditions required to set those moments to zero once $M_0(\Lambda; x)$ has been set to 1.

We prove the pattern in the table by performing the count of the number of independent conditions. We calculate the coefficients of $N_p(\Lambda; x)$ for $p = 0, \ldots, \ell$ and show that, because of the regularity and symmetry of the kernel, exactly $\lfloor \ell/2 \rfloor + 1$ of those coefficients are independent of one another.

It is convenient to express regularity at the nodes in terms of the coefficients of the constituent polynomials as defined in Eq. (2). The regularity conditions are $P_k^{(s)}(k+1) = P_{k+1}^{(s)}(k+1)$ for $0 \le k \le R-1$ and $0 \le s \le \ell - 1$. In terms of the coefficients we have

$$\sum_{j=s}^{\ell} \binom{j}{s} a_{k,j} = a_{k+1,s}. \tag{3}$$

The following Lemma will help us prove subsequent results.

Lemma 1. *Let $\Lambda \in S_R^\ell$ be defined for $x \in [k, k+1)$ as*

$$\Lambda(x) = P_k(x) = \sum_{n=0}^{\ell} a_{k,n} \; (x-k)^n.$$

Then, for $m \le \ell$, the conditions $N_p(\Lambda; z) = z^p$ for $z = 0, 1, \ldots, m$ are equivalent to the matrix product $\mathcal{VA}$ being equal to the first $m+1$ rows of the $(\ell+1) \times (\ell+1)$ identity matrix. Here, $\mathcal{V}$ is an $(m+1) \times 2R$ matrix of Vandermonde type given by

$$\mathcal{V}_{i,j} = \begin{cases} (j-R)^{i-1} & \text{for} \quad (i,j) \neq (1,R) \\ 1 & \text{for} \quad (i,j) = (1,R), \end{cases}$$

and $\mathcal{A}$ is the matrix of coefficients of size $2R \times (\ell+1)$ given by

$$\mathcal{A}_{j,k} = a_{R-j,k-1}.$$

Proof. Let $0 \le z < 1$, then for $p = 0, 1, \ldots, m$ we have

$$N_p(\Lambda; z) = \sum_{k=1-R}^{R} k^p \Lambda(z-k) = \sum_{k=1-R}^{R} k^p P_{-k}(z-k) = \sum_{n=0}^{\ell} \left(\sum_{k=1-R}^{R} k^p \, a_{-k,n} \right) z^n$$

which can be written as the matrix product

$$\begin{pmatrix} (1-R)^p & (2-R)^p & \cdots & 0^p & 1^p & \cdots & R^p \end{pmatrix} \begin{pmatrix} a_{R-1,0} & a_{R-1,1} & \cdots & a_{R-1,\ell} \\ a_{R-2,0} & a_{R-2,1} & \cdots & a_{R-2,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{-R,0} & a_{-R,1} & \cdots & a_{-R,\ell} \end{pmatrix} \begin{pmatrix} 1 \\ z \\ \vdots \\ z^\ell \end{pmatrix}.$$

This is precisely the $(p+1)^{st}$ row of $\mathcal{VA}$ multiplied by the vector $\vec{z} = (1, z, z^2, \ldots, z^\ell)^T$. □

We are now ready to state the theorem that shows that the pattern observed in Table 1 holds for all ℓ.

Theorem 1. *Fix $\ell \ge 1$. If $\Lambda \in S_R^\ell \cap C^{\ell-1}(\mathbb{R})$ is an even function, then*

1. *$M_0 = 1$ represents 1 condition.*
2. *The number of additional conditions for $M_k(\Lambda; x) = 0$ for $k = 1, \ldots, \ell$ is*

$$\begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}$$

3. *$M_{\ell+1}(\Lambda; x) \neq 0$.*
4. *The minimum support for $O(h^{\ell+1})$ interpolation is $R = 2\lfloor \ell/2 \rfloor + 1$ and there is a unique kernel with this support.*

Proof. The strategy for the proof is to determine the implications of the regularity and symmetry of Λ on the matrix of coefficients $\mathcal{A}$ and use Lemma 1.

Regularity: Since $\Lambda(x) \in C^{\ell-1}(\mathbb{R})$, we have that

$$P_k^{(s)}(k+1) = P_{k+1}^{(s)}(k+1) \quad \text{for} \quad s = 0, 1, \ldots, \ell-1$$

for all k. For convenience we will use the fact that all coefficients of $P_R(x)$ and $P_{-R-1}(x)$ are zero since they lie outside the support of the kernel. In terms of the polynomial coefficients, the regularity conditions are

$$\sum_{i=s}^{\ell} \binom{i}{s} a_{k,i} = a_{k+1,s} \quad \text{for} \quad 0 \le s \le \ell - 1 \quad \text{and} \quad -R-1 \le k \le R-1. \tag{4}$$

In particular, the case $k = -R-1$ implies that $a_{-R}, i = 0$ for $i = 0, 1, \ldots, \ell-1$ so that the leftmost polynomial is $P_{-R}(x) = a_{-R,\ell}(x+R)^\ell$.

Symmetry: For an even kernel $\Lambda(x)$, we require that $\Lambda(-x) = \Lambda(x)$. That is, for $x \in (k, k+1)$, $P_k(x) = P_{-k-1}(-x)$ which is

$$\begin{aligned}\sum_{n=0}^{\ell} a_{k,n}\ (x-k)^n &= \sum_{n=0}^{\ell} a_{-k-1,n}\ [1-(x-k)]^n \\ &= \sum_{n=0}^{\ell} \left[\sum_{j=n}^{\ell} (-1)^n \binom{j}{n} a_{-k-1,j}\right] (x-k)^n.\end{aligned}$$

Since this equation must hold for all $x \in (k, k+1)$, we have that

$$a_{k,n} = (-1)^n \sum_{j=n}^{\ell} \binom{j}{n} a_{-k-1,j} \quad \text{for} \quad 0 \le n \le \ell \quad \text{and} \quad -R \le k \le R-1. \tag{5}$$

Substituting $i = -k-1$ in Eq. (4) and combining with Eq. (5) we conclude that

$$a_{i,n} = (-1)^n\ a_{-i,n} \quad \text{for} \quad 0 \le n \le \ell - 1 \quad \text{and} \quad 0 \le i \le R. \tag{6}$$

If we define for $n = 0, 1, \ldots, \ell$ and $m \ge 1$

$$B_{0,n} = \sum_{k=1-R}^{R} a_{-k,n} \quad \text{and} \quad B_{m,n} = \sum_{k=1-R}^{R} k^m\ a_{-k,n}$$

then proving $\mathcal{VA} = \mathcal{I}$ reduces to showing that $B_{m,n} = \delta_{m,n}$ for $0 \le m, n \le \ell$. Notice, in particular, that Eq. (6) shows that $B_{2j+1,0} = 0$ for $j = 0, 1, \ldots$.

Taking Eq. (4) and summing over k we get that

$$\sum_{n=s+1}^{\ell} \binom{n}{s} B_{0,n} = 0 \tag{7}$$

$$\sum_{n=s+1}^{\ell} \binom{n}{s} B_{m,n} = \sum_{j=0}^{m-1} \binom{m}{j} B_{j,s} \tag{8}$$

for $0 \le s \le \ell - 1$ and $1 \le m \le \ell$.

Eq. (7) for $0 \le s \le \ell - 1$ is a homogeneous invertible triangular system for $B_{0,1}, B_{0,2}, \ldots, B_{0,\ell}$ so that $B_{0,k} = 0$ for $k = 1, 2, \ldots, \ell$. Note that $B_{0,0}$ is not determined. Since the condition $M_0(\Lambda; z) = N_0(\Lambda; z) = 1$ is equivalent to $B_{0,k} = \delta_{0,k}$ (the first row of the identity matrix), then $M_0(\Lambda; z) = 1$ is achieved by the single condition $B_{0,0} = 1$. This proves part 1.

We proceed by induction. Assume $m \le \ell$ and $B_{j,s} = \delta_{j,s}$ for $0 \le j \le m-1$ and $0 \le s \le \ell$. Then Eq. (8) is

$$\sum_{n=s+1}^{\ell} \binom{n}{s} B_{m,n} = \sum_{j=0}^{m-1} \binom{m}{j} \delta_{j,s} = \begin{cases} \binom{m}{s} & \text{for} \quad 0 \le s \le m-1 \\ 0 & \text{for} \quad m \le s \le \ell - 1. \end{cases}$$

Therefore, the variables $B_{m,k}$ for $k = m+1, m+2, \ldots, \ell$ satisfy the same invertible homogeneous system and thus $B_{m,k} = 0$ for $k = m+1, m+2, \ldots, \ell$. Then Eq. (8) reduces to

$$\sum_{n=s+1}^{m} \binom{n}{s} B_{m,n} = \binom{m}{s} \quad \text{for} \quad 0 \le s \le m-1.$$

Using back substitution, when $s = m-1$ we have that $B_{m,m} = 1$, which further reduces Eq. (8) to

$$\sum_{n=s+1}^{m-1} \binom{n}{s} B_{m,n} = 0 \quad \text{for} \quad 0 \le s \le m-2.$$

This invertible homogeneous system results in $B_{m,k} = 0$ for $k = 1, 2, \ldots, m-1$. Thus, all entries of the m-th row of the matrix $\mathcal{VA}$, except $B_{m,0}$, are determined and agree with the m-th row of the identity matrix. Since $B_{m,0} = 0$ for m odd, the odd moments are satisfied without additional constraints. Each even moment, $m = 2j$ ($j \ge 1$), requires the single condition $B_{2j,0} = 0$. This argument holds for $1 \le m \le \ell$. This proves part 2.

To see that $M_{\ell+1}(\Lambda; z) \ne 0$ we check that $N_{\ell+1}(\Lambda; z) \ne z^{\ell+1}$. This is easy to see since $N_{\ell+1}(\Lambda; z)$ is a polynomial of degree ℓ.

Finally, as mentioned earlier, there are $R - \lfloor \ell/2 \rfloor$ degrees of freedom left after imposing regularity and symmetry (evenness). Since the degrees of freedom are used to set $B_{2j,0} = \delta_{j,0}$ for $j = 0, 1, \ldots, \lfloor \ell/2 \rfloor$, we must have $R \ge 2\lfloor \ell/2 \rfloor + 1$. For $R = 2\lfloor \ell/2 \rfloor + 1$, uniqueness follows from the unique solution for the $B_{m,n}$'s. □

As a result of this theorem, we conclude that given $\ell > 0$ and $R = 2\lfloor \ell/2 \rfloor + 1$, there exists a unique kernel in $S_R^\ell \cap C^{\ell-1}(\mathbb{R})$ which interpolates to $O(h^{\ell+1})$. There are more kernels in S_R^ℓ for $R > 2\lfloor \ell/2 \rfloor + 1$ but they are supported over larger intervals without any gain of accuracy. For $\ell = 1$, the unique smooth kernel is simply the hat function. The unique smooth kernels, Λ_ℓ^{sm}, for $\ell = 2, 3$, and 4 are displayed below (the even extension is implied) and their plots are shown in Figure 1.

$$\Lambda_2^{sm}(x) = \begin{cases} \frac{1}{8}(5 - 3\,x^2) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{16}(23 - 26\,x + 7\,x^2) & \text{for} \quad 1 \le x < 2 \\ \frac{-1}{16}(3 - x)^2 & \text{for} \quad 2 \le x < 3 \\ 0 & \text{for} \quad 3 \le x \end{cases}$$

$$\Lambda_3^{sm}(x) = \begin{cases} \frac{1}{18}(15 - 27\,x^2 + 14\,x^3) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{36}(69 - 117\,x + 63\,x^2 - 11\,x^3) & \text{for} \quad 1 \le x < 2 \\ \frac{1}{36}(-3 + x)^3 & \text{for} \quad 2 \le x < 3 \\ 0 & \text{for} \quad 3 \le x \end{cases}$$

$$\Lambda_4^{sm}(x) = \begin{cases} \frac{1}{3456}(2311 - 1830\,x^2 + 355\,x^4) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{1728}(671 + 1938\,x - 3822\,x^2 + 1938\,x^3 - 307\,x^4) & \text{for} \quad 1 \le x < 2 \\ \frac{1}{1728}(8159 - 13038\,x + 7410\,x^2 - 1806\,x^3 + 161\,x^4) & \text{for} \quad 2 \le x < 3 \\ \frac{1}{6912}(-30787 + 32412\,x - 12642\,x^2 + 2172\,x^3 - 139\,x^4) & \text{for} \quad 3 \le x < 4 \\ \frac{13}{6912}(-5 + x)^4 & \text{for} \quad 4 \le x < 5 \\ 0 & \text{for} \quad 5 \le x \end{cases}$$

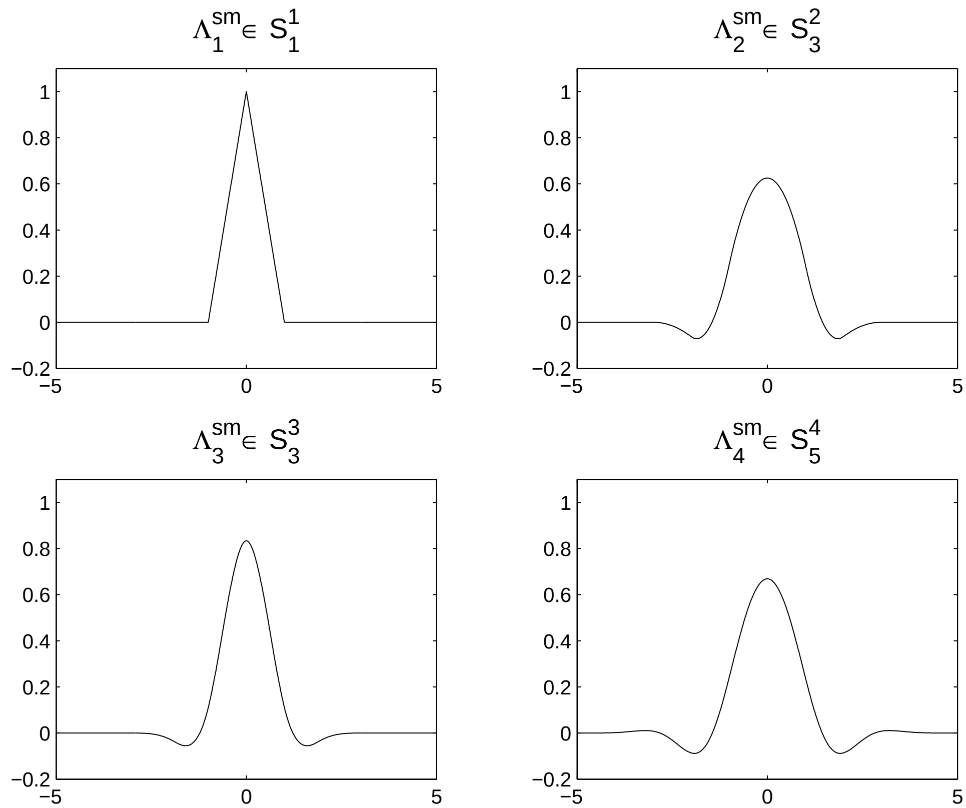

Figure 1: Examples of smooth even kernels.

3.2 Narrow Kernels

We now consider designing kernels in S_R^ℓ without imposing any regularity conditions. Instead, we use all degrees of freedom to satisfy as many moment conditions as possible. This will result in a highly accurate interpolation kernel with relatively small support. The correct way to pose the relevant questions is to fix the size of the support R and ask what is the maximum number of moment conditions that can be satisfied as well as the minimum polynomial degree that can accomplish it.

Theorem 2. *Fix $R \geq 1$. The maximum order of accuracy which can be obtained from any interpolation kernel in any of the spaces S_R^ℓ is $2R$. That order is obtained by a unique, even kernel $\Lambda \in S_R^{2R-1} \cap C^0(\mathbb{R})$.*

Proof. A piecewise polynomial kernel that interpolates a smooth function with $O(h^q)$ accuracy must satisfy $N_p(\Lambda; x) = x^p$ for $p = 0, \dots, q-1$. From Lemma 1, we know this is equivalent to $\mathcal{V}\mathcal{A}$ being equal to the first q rows of the identity matrix $\mathcal{I}$ of size $\ell + 1$. The Vandermonde matrix $\mathcal{V}$ is $q \times 2R$, the coefficient matrix $\mathcal{A}$ is $2R \times (\ell + 1)$ and both are defined in the Lemma.

If $q = 2R$, the Vandermonde matrix $\mathcal{V}$ is square and invertible; therefore, all coefficients in $\mathcal{A}$ are uniquely determined and the interpolation has order of accuracy $2R$. Notice that if $\ell \geq 2R$, the columns of $\mathcal{A}$ numbered $2R+1$ through $\ell+1$ must be zero since they equal $\mathcal{V}^{-1}$ times the first $2R$ rows of the identity matrix of size $\ell + 1$. Since $\mathcal{A}$ contains the polynomial coefficients, this implies that the maximum polynomial degree is $2R - 1$. If $\ell = 2R - 1$, the unique solution for $\mathcal{A}$ contains at least one polynomial of degree $2R - 1$. Thus $\Lambda \in S_R^{2R-1}$. If $q \geq 2R + 1$, then by considering the first $2R$ equations and the argument above, we see that P_k has degree at most $2R - 1$ for each k, so $N_{q-1}(\Lambda; x)$ cannot equal x^{q-1}. Therefore the maximum possible order of accuracy is $q = 2R$.

It remains to show that $\Lambda(x) \in S_R^{2R-1}$ of order $O(h^{2R})$ is continuous and even. In this case, $\mathcal{V}\mathcal{A} = \mathcal{I}$, so that $\mathcal{A} = \mathcal{V}^{-1}$.

Let $x_j = j - R$, for $j = 1, \dots, 2R$. Then $\mathcal{V}_{i,j} = x_j^{i-1}$. A simple observation (see [15] §4.6) shows that the inverse of $\mathcal{V}$ is given in terms the coefficients of a polynomial interpolation problem. Specifically, let L_i be the Lagrange polynomial

$$L_i(x) = \prod_{k \neq i} \frac{x - x_k}{x_i - x_k}.$$

Then, for $1 \leq i, k \leq 2R$, $L_i(x_j) = \delta_{i,j}$. Define the matrix $\mathcal{C}$ by

$$L_i(x) = \sum_{n=1}^{2R} \mathcal{C}_{i,n}\ x^{n-1}.$$

Then the identity $L_i(x_j) = \delta_{i,j}$ means exactly $\mathcal{V}\mathcal{C} = \mathcal{I}$, since

$$\delta_{i,j} = L_i(x_j) = \sum_{n=1}^{2R} \mathcal{C}_{i,n}\ x_j^{n-1} = \sum_{n=1}^{2R} \mathcal{C}_{i,n}\mathcal{V}_{n,j}.$$

Thus, $\mathcal{C}_{i,j} = a_{R-i,j-1}$, and

$$L_i(x) = \sum_{n=0}^{2R-1} a_{R-i,n}\ x^n. \tag{9}$$

The symmetry and continuity of Λ follow from two properties of the Lagrange polynomials:

1. $L_1(1) = L_{2R}(0) = 0$ and $L_i(1) = L_{i-1}(0)$, for $i = 2, ..., 2R$,
2. $L_i(x) = L_j(1-x)$, if $i + j = 2R + 1$.

The first property can be seen by noting that $x_{R+1} = 1$, $x_R = 0$, and $L_1(x_{R+1}) = L_{2R}(x_R) = 0$. Also, for $2 \le i \le 2R$,

$$L_i(x_{R+1}) = \delta_{i,R+1} = \delta_{i-1,R} = L_{i-1}(x_R).$$

The last equation and Eq. (9) imply that

$$\sum_{n=0}^{2R-1} a_{R-1,n} = a_{-R,0} = 0, \quad \text{and} \quad \sum_{n=0}^{2R-1} a_{R-i,n} = a_{R+1-i,0}, \quad \text{for} \quad i = 2, ..., 2R.$$

These identities mean exactly $\Lambda \in C^0(\mathbb{R})$.

The second property follows from a substitution in the product which defines L_i. Let $j = 2R+1-i$ and $m = 2R+1-k$.

$$\begin{aligned} L_i(x) = \prod_{k \ne i} \frac{x - x_k}{x_i - x_k} &= \prod_{k \ne i} \frac{x + R - k}{i - k} \\ &= \prod_{m \ne 2R+1-i} \frac{x + R + m - 2R - 1}{i + m - 2R - 1} \\ &= \prod_{m \ne j} \frac{1 - x + R - m}{j - m} \\ &= \prod_{m \ne j} \frac{1 - x - x_m}{x_j - x_m} \\ &= L_j(1-x) \end{aligned}$$

Take $1 \le i \le R$, and $j = 2R + 1 - i$. Differentiating d times and evaluating at $x = 0$ gives $L_i^{(d)}(0) = (-1)^d L_j^{(d)}(1)$. That is, for $1 \le i \le R$,

$$a_{R-i,d} = (-1)^d \sum_{n=d}^{2R-1} \binom{n}{d} a_{R-j,n}.$$

Using $j = 2R + 1 - i$, we find that the last equation reduces to

$$a_{-k,d} = (-1)^d \sum_{n=d}^{2R-1} \binom{n}{d} a_{k-1,n}$$

for $1 \le k \le R$. But, this last identity means exactly $P_{-k}(x - k) = P_{k-1}(k - x)$, for $1 \le k \le R$. Therefore, Λ is even.

$\square$

One can check that the first derivative of these kernels is discontinuous, so that they do not have more regularity than continuity. We note that the last part of the proof shows not only that Λ is continuous but that it is zero at each nonzero integer. This implies that the polynomials in Λ are easily factored as shown in the examples below (the even extension is implied). Figure 2 shows the plots of these functions. The hat function is also a narrow kernel in the sense discussed here. The hat function and $\Lambda_2^{narr}(x)$ have been analyzed in [29].

$$\Lambda_2^{narr}(x) = \begin{cases} \frac{1}{2}(1+x)(1-x)(2-x) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{6}(1-x)(2-x)(3-x) & \text{for} \quad 1 \le x < 2 \\ 0 & \text{for} \quad 2 \le x \end{cases}$$

$$\Lambda_3^{narr}(x) = \begin{cases} \frac{1}{12}(2+x)(1+x)(1-x)(2-x)(3-x) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{24}(1+x)(1-x)(2-x)(3-x)(4-x) & \text{for} \quad 1 \le x < 2 \\ \frac{1}{120}(1-x)(2-x)(3-x)(4-x)(5-x) & \text{for} \quad 2 \le x < 3 \\ 0 & \text{for} \quad 3 \le x \end{cases}$$

$$\Lambda_4^{narr}(x) = \begin{cases} \frac{1}{144}(3+x)(2+x)(1+x)(1-x)(2-x)(3-x)(4-x) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{240}(2+x)(1+x)(1-x)(2-x)(3-x)(4-x)(5-x) & \text{for} \quad 1 \le x < 2 \\ \frac{1}{720}(1+x)(1-x)(2-x)(3-x)(4-x)(5-x)(6-x) & \text{for} \quad 2 \le x < 3 \\ \frac{1}{5040}(1-x)(2-x)(3-x)(4-x)(5-x)(6-x)(7-x) & \text{for} \quad 3 \le x < 4 \\ 0 & \text{for} \quad 4 \le x \end{cases}$$

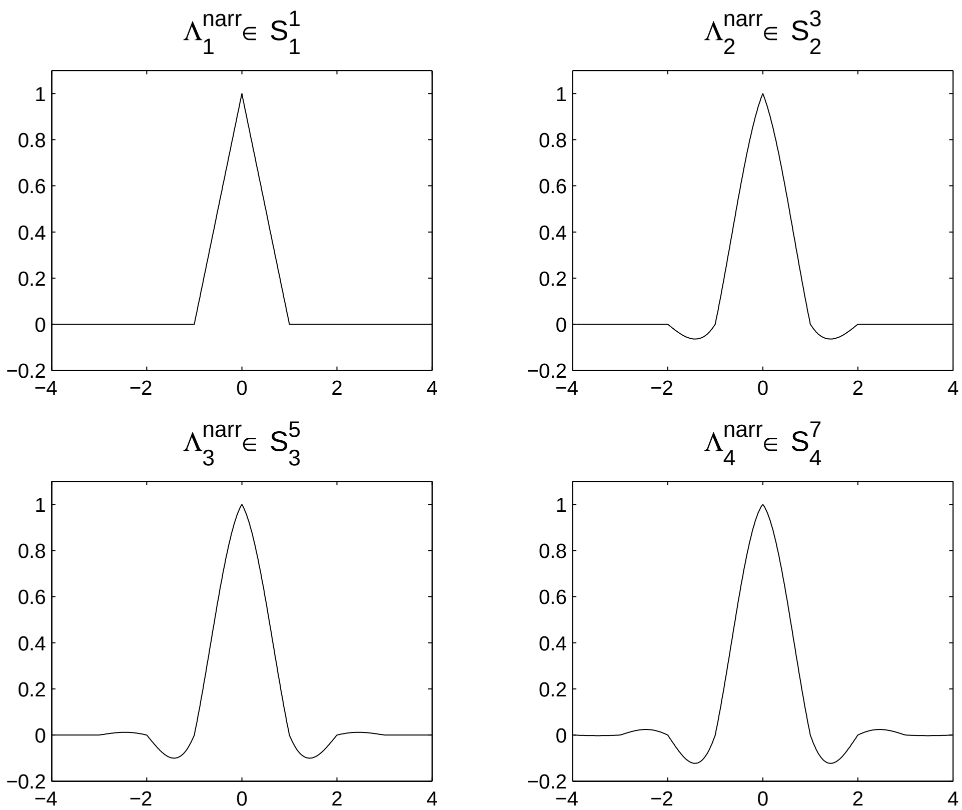

Figure 2: Examples of narrow even kernels.

Note that the narrow kernels have a form that can be written in a compact way. Given R, then for

$k = 0, 1, \ldots, R-1$ we have that $\Lambda_R^{narr}(x) = P_k(x)$ for $x \in [k, k+1)$, where

$$P_k(x) = -\prod_{\substack{n=k+1-R \\ n\neq 0}}^{k+R} \frac{1}{n}(x-n).$$

We conclude this section with a summary in Table 2 of the two types of kernels. We emphasize that for a given polynomial degree and order of accuracy, there is a tradeoff between the regularity and the size of the support of the kernels.

smooth kernels	**narrow kernels**
Given ℓ: $R = 2\lfloor \ell/2 \rfloor + 1$, $O(h^{\ell+1})$, $C^{\ell-1}(\mathbb{R})$	Given R: $\ell = 2R-1$, $O(h^{2R})$, $C^0(\mathbb{R})$
$\ell = 1,\ R = 1,\ O(h^2),\ C^0(\mathbb{R})$	$\ell = 1,\ R = 1,\ O(h^2),\ C^0(\mathbb{R})$
$\ell = 2,\ R = 3,\ O(h^3),\ C^1(\mathbb{R})$	
$\ell = 3,\ R = 3,\ O(h^4),\ C^2(\mathbb{R})$	$\ell = 3,\ R = 2,\ O(h^4),\ C^0(\mathbb{R})$
$\ell = 4,\ R = 5,\ O(h^5),\ C^3(\mathbb{R})$	
$\ell = 5,\ R = 5,\ O(h^6),\ C^4(\mathbb{R})$	$\ell = 5,\ R = 3,\ O(h^6),\ C^0(\mathbb{R})$

Table 2: Summary and a few examples of the two classes of unique kernels.

4 Comments about the errors

For a smooth function $f(x)$, the leading error term in the interpolation $(L_0 f)(x)$ of order n is

$$(-1)^n\ h^n\ \frac{1}{n!} f^{(n)}(x)\ M_n(\Lambda; x).$$

Sometimes this error displays oscillations in a scale comparable to the grid size h. It can be seen that for $0 \le x < h$, we may assume that $f^{(n)}(x)$ is slowly varying and so the character of the error comes from $M_n(\Lambda; x)$ which is the first nonvanishing moment of the kernel Λ. For the smooth kernels discussed in Section 3.1, these are

$$\begin{aligned}
\ell = 1: &\quad M_2(\Lambda; x) = -x(x-1) \\
\ell = 2: &\quad M_3(\Lambda; x) = x(x-1/2)(x-1) \\
\ell = 3: &\quad M_4(\Lambda; x) = -x^2(x-1)^2 \\
\ell = 4: &\quad M_5(\Lambda; x) = x(x-1/2)(x-1)(x^2-x-1/3).
\end{aligned}$$

Figure 3 shows the errors on a very fine scale resulting from the first example in Section 7. These are typical in interpolation computations. The error in the top graph results from using the hat function ($\ell = 1$) and the error in the bottom graph results from using $\Lambda_2^{sm}(x)$ ($\ell = 2$). Notice that they follow the pattern in the equations above. The period of the oscillations is h so that a plot of the error on a fine scale will look noisy. In principle, one can use one degree of freedom in the design of the kernel to set the leading error moment to a nonzero constant in order to remove the oscillations. However, this would result in in fewer moment conditions being satisfied for a given support R.

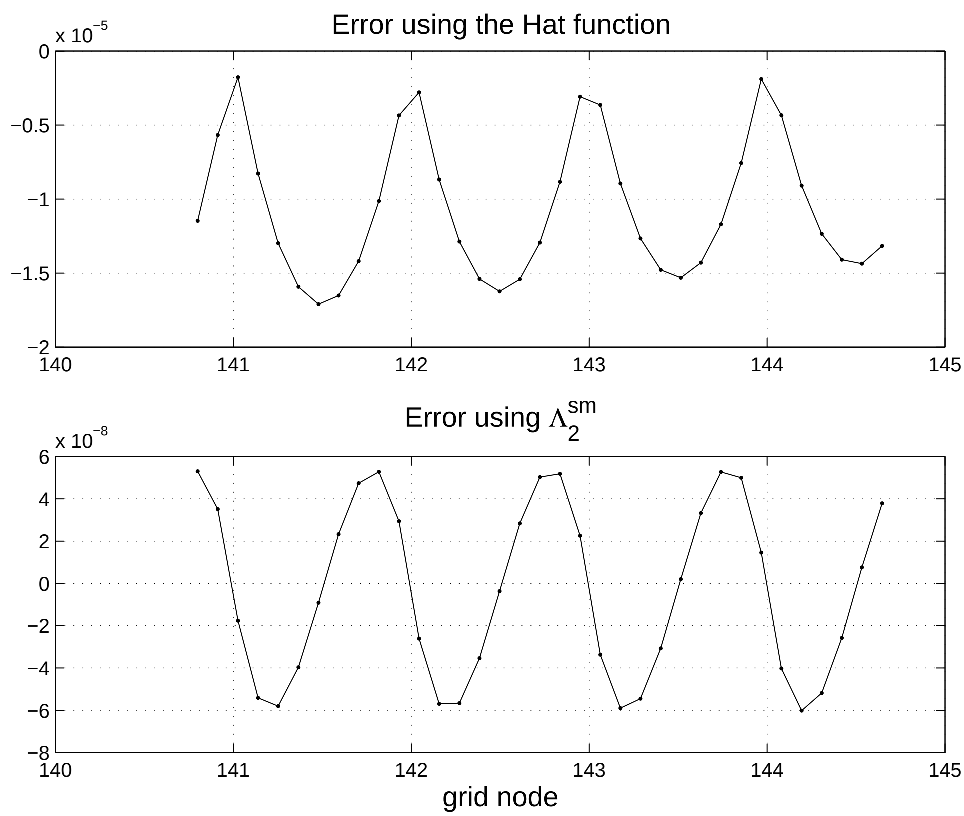

Figure 3: Errors in a typical interpolation computation. The top graph shows the error from the hat function ($\ell = 1$) and the bottom graph shows the error from $\Lambda_2^{sm}(x)$ ($\ell = 2$).

5 Dilations of kernels

In numerical applications, kernels of the type shown in Figures 1 and 2 are used as smooth approximations of a delta distribution to interpolate data from arbitrary points to the grid nodes. The convergence properties of such methods typically depend on the relative size of the grid to the radius of the support of the kernel, so that convergence may be achieved as the grid size and the support of the kernel approach zero at different rates [16, 10, 7, 12, 17]. So far, we have designed our kernels so that the size of their support, R, is proportional to h. However, it may be important to relax this condition for the reasons mentioned above.

In this section we show that a kernel with support $[-R, R]$ can be scaled to have support $[-nR, nR]$ for any positive integer n. This provides some flexibility for reducing the support of the kernel and the grid size at different rates, although the support must always remain a multiple of R and cannot be scaled independently of h (see also [28]).

Proposition 2. *Given Λ with support $[-R, R]$, assume that $(L_s f)(x)$ approximates $f^{(s)}(x)$ so that $M_p(\Lambda; x) = (-1)^s\ s!\ \delta_{p,s}$ for $p = 0, 1, \ldots, q$ with $0 \le s < q$. Define the scaled dilation of Λ by*

$$\Phi(x) = \frac{1}{n^{s+1}}\Lambda\left(\frac{x}{n}\right)$$

for some positive integer n. Let $0 \le x < 1$, then

$$M_p(\Phi; x) = M_p(\Lambda; x) \qquad \text{for} \quad p = 0, 1, \ldots, q.$$

Proof. Note that Φ has support $[-nR, nR]$ on the same grid of size 1. For $p = 0, 1, \ldots, q$, we have

$$\begin{aligned} M_p(\Phi; x) &= \sum_{m=1-nR}^{nR} (x-m)^p\ \Phi(x-m) \\ &= n^{p-s-1} \sum_{m=1-nR}^{nR} \left(\frac{x}{n} - \frac{m}{n}\right)^p\ \Lambda\left(\frac{x}{n} - \frac{m}{n}\right). \end{aligned}$$

Notice that $0 \le x/n < 1/n$ so that the points $z_j = x/n + j/n$ satisfy $0 \le z_j < 1$ for $j = 0, 1, \ldots, n-1$, therefore we can break up the sum into

$$\begin{aligned} M_p(\Phi; x) &= n^{p-s-1} \sum_{k=1-R}^{R} \sum_{j=0}^{n-1} (z_j - k)^p\ \Lambda(z_j - k) \\ &= n^{p-s-1} \sum_{j=0}^{n-1} M_p(\Lambda; z_j) \\ &= n^{p-s}(-1)^s\ s!\ \delta_{p,s} \\ &= M_p(\Lambda; x) \qquad \text{for} \ \ p = 0, 1, \ldots, q. \end{aligned}$$

□

The preceding result shows that the regularity and moment conditions satisfied by the kernel Λ are also satisfied by Φ. They both have the same polynomial pieces (scaled by n) except that the support of Φ is larger than that of Λ by a factor of n.

Perhaps the most common example of this type of scaling is the hat function, $H(x)$, and the wide-hat function, $W(x)$ with $n = 2$:

$$H(x) = \begin{cases} 1 - |x| & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \geq 1, \end{cases} \qquad W(x) = \begin{cases} \frac{1}{4}(2 - |x|) & \text{for} \quad |x| < 2 \\ 0 & \text{for} \quad |x| \geq 2. \end{cases}$$

6 Differentiation of Interpolation Kernels

In this section, we discuss properties of derivatives of the piecewise polynomial kernels in the previous sections. Since each polynomial $P_k(x)$ is defined for $x \in [k, k+1)$, the kernels are right-differentiable infinitely many times. Therefore, we use the notation

$$\frac{d}{dx^+}\Lambda(x) = \Lambda'(x) = \lim_{\epsilon \to 0^+} \frac{\Lambda(x+\epsilon) - \Lambda(x)}{\epsilon}.$$

An important recurrence can be found by differentiation of the moments of a kernel.

$$\begin{aligned} M_0'(\Lambda; x) &= \frac{d}{dx^+}\left[\sum_k \Lambda(x-k)\right] \\ &= \sum_k \Lambda'(x-k) \\ &= M_0(\Lambda'; x) \end{aligned}$$

and for $n \geq 1$

$$\begin{aligned} M_n'(\Lambda; x) &= \frac{d}{dx^+}\left[\sum_k (x-k)^n \Lambda(x-k)\right] \\ &= \sum_k n(x-k)^{n-1}\Lambda(x-k) + (x-k)^n \Lambda'(x-k) \\ &= nM_{n-1}(\Lambda; x) + M_n(\Lambda'; x). \end{aligned}$$

Solving for $M_n(\Lambda'; x)$ we get the recursion

$$M_0(\Lambda'; x) = M_0'(\Lambda; x) \tag{10}$$

$$M_n(\Lambda'; x) = M_n'(\Lambda; x) - nM_{n-1}(\Lambda; x) \quad \text{for } n \geq 1. \tag{11}$$

Now consider a kernel Λ which interpolates to $O(h^{p+1})$. Then

$$M_n(\Lambda; x) = \delta_{0,n} \ \text{ for } 0 \leq n \leq p \qquad \text{and} \qquad M_{p+1}(\Lambda; x) = g(x) \neq 0.$$

By using the recursion in Eq. (10)–(11), we find that

$$M_n(\Lambda'; x) = -\delta_{1,n} \ \text{ for } 0 \leq n \leq p \qquad \text{and} \qquad M_{p+1}(\Lambda'; x) = g'(x) - p\delta_{0,p-1} \neq 0.$$

Thus, Λ' is a kernel that can be used to approximate $f'(x)$. In general, the approximation is $O(h^p)$, although it improves to $O(h^{p+1})$ when $g'(x) = p\delta_{0,p-1}$ but this is not usually the case as discussed in Section 4. The recursion in Eq. (10)–(11) can be used for higher derivatives of the kernel, leading to the following result.

Lemma 2. *Let $\Lambda \in S_R^\ell$ and Λ' be the right-differentiation of $\Lambda(x)$. If the kernel Λ approximates the d^{th} derivative of smooth functions with order of accuracy $O(h^p)$, then Λ' approximates the $(d+1)^{st}$ derivative to at least $O(h^{p-1})$.*

6.1 Smooth kernels for approximating $f'(x)$

One may use the previous lemma to construct kernels. For example, to approximate the d-th derivative of some function with order of accuracy p, we could construct an order $(p+d)$ interpolation kernel and then differentiate it d times. However, this process does not always lead to kernels with the smallest support possible because differentiating the kernel lowers the degree and the accuracy but does not decrease the support.

An alternative to differentiating smooth interpolation kernels to derive kernels for approximating $f'(x)$ is to design them with the desired properties from the start. We let $\Lambda(x)$ be an odd piecewise polynomial kernel of degree ℓ, enforce the regularity conditions in Eq. (3) and set $M_j(\Lambda; x) = -\delta_{j,1}$ for $j = 0, 1, \ldots, m$. This results in an $O(h^m)$ approximation of $f'(x)$.

Table 3 summarizes the findings for polynomials of degree $\ell = 1, \ldots, 8$. The row labeled M_0 gives the number of conditions needed to enforce $M_0(\Lambda; x) = 0$. Notice that this is automatically satisfied due to regularity and symmetry. The other rows show the number of conditions required to set those moments to their corresponding value. The line in a cell indicates that the corresponding moment cannot be set to zero, regardless of the size of the support R. In all cases, the resulting unique kernel is in $S_R^\ell \cap C^{\ell-1}(\mathbb{R})$ and the minimum support is $R = \ell + 1$.

Smooth Odd Kernels

	ℓ							
	1	2	3	4	5	6	7	8
M_0	0	0	0	0	0	0	0	0
M_1	1	1	1	1	1	1	1	1
M_2	0	0	0	0	0	0	0	0
M_3	—	1	1	1	1	1	1	1
M_4		—	0	0	0	0	0	0
M_5			—	1	1	1	1	1
M_6				—	0	0	0	0
M_7					—	1	1	1
M_8						—	0	0
M_9							—	1
M_{10}								—

Table 3: The row labeled M_0 gives the number of conditions needed to enforce $M_0(\Lambda; x) = 0$. The other rows show the number of conditions required to set those moments to their corresponding value.

Notice that the table demonstrates that some, but not all, of these kernels are the derivative of one of the smooth interpolation kernels discussed earlier. Suppose Φ is an interpolation kernel in $S_R^{\ell+1} \cap C^\ell(\mathbb{R})$ and therefore has support $R = 2\lfloor \frac{\ell+1}{2} \rfloor + 1$ and order of accuracy $O(h^{\ell+2})$. Then we know that Φ' is an approximation kernel (for f') in $S_R^\ell \cap C^{\ell-1}(\mathbb{R})$ with order of accuracy $O(h^{\ell+1})$. If ℓ is odd, then $R = 2\lfloor \frac{\ell+1}{2} \rfloor + 1 = \ell + 2$ which is larger than the support of the kernels in Table 3. Therefore Φ' is not one with minimum support. On the other hand, if ℓ is even, then $R = 2\lfloor \frac{\ell+1}{2} \rfloor + 1 = \ell + 1$ and Φ' must be one of the kernels in Table 3.

Theorem 3. *Fix $\ell \geq 1$. If $\Lambda \in S_R^\ell \cap C^{\ell-1}\mathbb{R}$ is an odd function, then*

1. *$M_0 = 1$ represents 0 conditions.*

2. *The number of additional conditions for $M_k(\Lambda; x) = 0$ for $k = 1, \ldots, \ell+1$ is*

$$\begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

3. $M_{\ell+2}(\Lambda; x) \neq 0$.

4. *The minimum support for $O(h^{\ell+1})$ approximation is $R = \ell+1$ and there is a unique kernel with this support.*

Proof. The proof is very similar to the proof of theorem 1 so we do not present all the details. The regularity arguments are as in theorem 1 so Eq. (4) holds. The symmetry condition for odd functions is $\Lambda(-x) = -\Lambda(x)$. This results in

$$a_{k,n} = (-1)^n \sum_{j=n}^{\ell} \binom{j}{n} a_{-k-1,j} \quad \text{for} \quad 0 \leq n \leq \ell \quad \text{and} \quad -R \leq k \leq R-1. \tag{12}$$

Substituting $i = -k-1$ in Eq. (4) and combining with Eq. (12) we conclude that

$$a_{i,n} = (-1)^{n+1}\, a_{-i,n} \quad \text{for} \quad 0 \leq n \leq \ell-1 \quad \text{and} \quad 0 \leq i \leq R. \tag{13}$$

In the case of an odd kernel to approximate $f'(x)$, the required conditions are that

$$\mathcal{V}\mathcal{A}\begin{pmatrix} 1 \\ z \\ \vdots \\ z^{\ell} \end{pmatrix} = \frac{d}{dz}\begin{pmatrix} 1 \\ z \\ \vdots \\ z^{\ell} \\ z^{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \ell z^{\ell-1} \\ (\ell+1)z^{\ell} \end{pmatrix}$$

which are equivalent to $B_{0,k} = 0$ and $B_{m,k} = m\ \delta_{k,m-1}$ for $0 \leq k \leq \ell$ and $1 \leq m \leq \ell+1$ (see Eq. (1)).

We note in this case that Eq. (13) shows that $B_{2j,0} = 0$ for $j = 0, 1, \ldots$. As in theorem 1, Eq. (7) for $0 \leq s \leq \ell-1$ is an invertible triangular system for the variables $B_{0,1}, B_{0,2}, \ldots, B_{0,\ell}$ so that $B_{0,k} = 0$ for $k = 1, 2, \ldots, \ell$. $B_{0,0}$ is already known to be zero. This proves part 1.

By induction, we assume $1 \leq m \leq \ell+1$ and $B_{j,s} = j\ \delta_{s,j-1}$ for $0 \leq j \leq m-1$ and $0 \leq s \leq \ell$. Then Eq. (8) is

$$\sum_{n=s+1}^{\ell} \binom{n}{s} B_{m,n} = \sum_{j=0}^{m-1} \binom{m}{j} j\ \delta_{s,j-1} = \begin{cases} \binom{m}{s+1}(s+1) & \text{for} \quad 0 \leq s \leq m-2 \\ 0 & \text{for} \quad m-1 \leq s \leq \ell-1. \end{cases}$$

Therefore, $B_{m,k} = 0$ for $k = m, m+1, \ldots, \ell$. Then Eq. (8) reduces to

$$\sum_{n=s+1}^{m-1} \binom{n}{s} B_{m,n} = \binom{m}{s+1}(s+1) = m\binom{m-1}{s} \quad \text{for} \quad 0 \leq s \leq m-2.$$

When $s = m-2$, we get that $B_{m,m-1} = m$. The rest of the variables satisfy $B_{m,k} = 0$ for $k = 1, 2, \ldots, m-2$. Since $B_{m,0} = 0$ for m even, the even moments are satisfied without additional constraints. Each odd moment, $m = 2j+1$ $(j \geq 0)$, requires the single condition $B_{2j+1,0} = 0$. The argument holds for $1 \leq m \leq \ell+1$. This proves part 2.

Since $N_{\ell+2}(\Lambda; z)$ is a polynomial of degree ℓ, it cannot equal $(\ell+2)z^{\ell+1}$; so $M_{\ell+2}(\Lambda; z) \neq 0$.

The symmetry condition for an odd function implies that the coefficients of the even powers of x in $P_0(x)$ must be zero. There are $\lfloor \frac{\ell+1}{2} \rfloor$ of them. Therefore, there are $R - \lfloor \frac{\ell+1}{2} \rfloor$ degrees of freedom available to satisfy $\lfloor \ell/2 \rfloor + 1$ moment conditions; that is, $R \geq \ell+1$. Uniqueness for $R = \ell+1$ follows. □

The theorem indicates that given ℓ, the maximum order of accuracy that can be attained is $O(h^{\ell+1})$ and that the minimum support to attain it is $R = \ell+1$. Moreover, there is a unique kernel that has this minimum support. The kernels corresponding to $\ell = 1, 2, 3$, and 4 are displayed below (the odd extension is implied).

$$\Lambda_1^{odd}(x) = \begin{cases} -\frac{1}{2}x & \text{for } 0 \leq x < 1 \\ \frac{1}{2}(x-2) & \text{for } 1 \leq x < 2 \\ 0 & \text{for } 2 \leq x \end{cases}$$

$$\Lambda_2^{odd}(x) = \begin{cases} \frac{1}{3}x(7x-9) & \text{for } 0 \leq x < 1 \\ \frac{1}{12}(-39+42x-11x^2) & \text{for } 1 \leq x < 2 \\ \frac{1}{12}(x-3)^2 & \text{for } 2 \leq x < 3 \\ 0 & \text{for } 3 \leq x \end{cases}$$

$$\Lambda_3^{odd}(x) = \begin{cases} \frac{1}{36}x(11x^2-30) & \text{for } 0 \leq x < 1 \\ \frac{1}{36}(29-117x+87x^2-18x^3) & \text{for } 1 \leq x < 2 \\ \frac{1}{36}(-179+195x-69x^2+8x^3) & \text{for } 2 \leq x < 3 \\ -\frac{1}{36}(x-4)^3 & \text{for } 3 \leq x < 4 \\ 0 & \text{for } 4 \leq x \end{cases}$$

$$\Lambda_4^{odd}(x) = \begin{cases} \frac{1}{2880}x(-6100+7100x^2-3069x^3) & \text{for } 0 \leq x < 1 \\ \frac{1}{2880}(4845-25480x+29070x^2-12280x^3+1776x^4) & \text{for } 1 \leq x < 2 \\ \frac{1}{2880}(-32595+49400x-27090x^2+6440x^3-564x^4) & \text{for } 2 \leq x < 3 \\ \frac{1}{5760}(40515-42140x+16290x^2-2780x^3+177x^4) & \text{for } 3 \leq x < 4 \\ -\frac{13}{5760}(x-5)^4 & \text{for } 4 \leq x < 5 \\ 0 & \text{for } 5 \leq x \end{cases}$$

These kernels are shown in Figure 4. Notice that $\Lambda_2^{odd}(x)$ is the derivative of $\Lambda_3^{sm}(x)$ in Section 3.1.

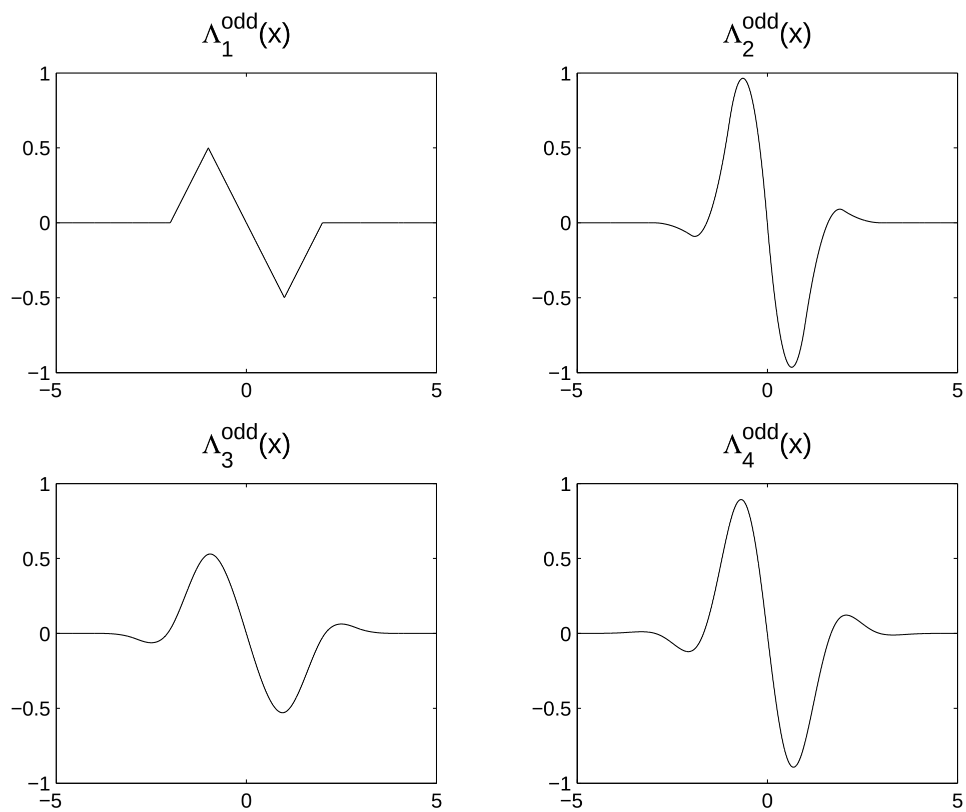

Figure 4: Examples of smooth odd kernels.

6.2 Narrow kernels for approximating $f'(x)$

The previous section shows that although all derivatives of smooth interpolation kernels satisfy the moment conditions for kernels that approximate the first derivative of smooth functions, the Λ' may not have the smallest support for the given properties. For narrow kernels, the situation is different. It turns out that for a given support radius R, the highest accuracy for approximating the first derivative of smooth functions is achieved by the derivative of the corresponding narrow interpolation kernel.

Theorem 4. *Fix $R \geq 1$. The maximum order of accuracy which can be obtained from any approximation kernel for the derivative of a smooth function in any of the spaces S_R^ℓ is $2R-1$. That order is obtained by a unique $\Lambda \in S_R^{2R-2}$ which is the derivative of the corresponding narrow interpolation kernel.*

Proof. A piecewise polynomial kernel that approximates the first derivative of a smooth function with $O(h^{q-1})$ accuracy must satisfy $N_p(\Lambda; x) = p\, x^{p-1}$ for $p = 0, \ldots, q-1$. This is equivalent to

$$\mathcal{VA}\begin{pmatrix}1\\ z\\ \vdots\\ z^\ell\end{pmatrix} = \begin{pmatrix}0\\ 1\\ \vdots\\ \ell z^{\ell-1}\\ (\ell+1)z^\ell\end{pmatrix} = \begin{pmatrix}0 & 0 & \cdots & 0\\ 1 & 0 & \cdots & 0\\ 0 & 2 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & \ell+1\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}1\\ z\\ \vdots\\ z^\ell\end{pmatrix} = \mathcal{J}\begin{pmatrix}1\\ z\\ \vdots\\ z^\ell\end{pmatrix}$$

where $\mathcal{V}$ is a $q \times 2R$ matrix of Vandermonde type, $\mathcal{A}$ is the coefficient matrix of size $2R \times (\ell+1)$ and $\mathcal{J}$ is $q \times (\ell+1)$. The matrices $\mathcal{V}$ and $\mathcal{A}$ are defined in Lemma 1.

If $q = 2R$, the Vandermonde matrix $\mathcal{V}$ is square and invertible; therefore, all coefficients in $\mathcal{A}$ are uniquely determined and the interpolation has order of accuracy $2R-1$. Notice that if $\ell+1 \geq 2R$, then the only the first $2R-1$ columns of $\mathcal{J}$ contain nonzero elements. Therefore, the columns of $\mathcal{A}$ numbered $2R$ through $\ell+1$ are zero and so the maximum polynomial degree is $2R-2$. When $\ell = 2R-2$, the unique solution for $\mathcal{A}$ contains at least one polynomial of degree $2R-2$. Thus $\Lambda \in S_R^{2R-2}$. If $q \geq 2R+1$, then by considering the first $2R$ equations and the argument above, we see that P_k has degree at most $2R-2$ for each k, so $N_{q-1}(\Lambda; x)$ cannot equal $(q-1)x^{q-2}$.

We note that if a narrow interpolation kernel $\Lambda \in S_R^{2R-1}$ has order of accuracy $2R$, then by Lemma 2, Λ' will have order of accuracy $2R-1$ and will belong to S_R^{2R-2}. □

The derivative of narrow kernels is discontinuous at the grid nodes. Using the derivative of a narrow kernel in the definition of $(L_1 f)(x)$ results in a discontinuous approximation of $f'(x)$. However, if this kernel is order $O(h^q)$, the discontinuity in the approximation $(L_1 f)(x)$ must be of the same order. In other words, the discontinuity is of the order of the error in the approximation; therefore, in theory these kernels are perfectly acceptable. In practice, however, there is a disadvantage when using discontinuous kernels that becomes apparent when approximating $f'(x)$ at a grid node. In that case, we must compute $\sum_k f_k \Lambda_h(-kh)h$ which requires evaluating the kernel exactly at the location of the discontinuities. Although Λ_h is defined properly at those points, the slightest bit of roundoff error can cause the evaluation of the kernel on the wrong side of the discontinuity, resulting in the wrong weights. The performance of these kernels will be illustrated in the numerical examples.

Narrow kernels have a curious property. Although their right derivative is discontinuous, the second right derivative is continuous. In fact, we have observed that the odd (nontrivial) right derivatives are discontinuous while the even right derivatives are continuous. Below are examples of two narrow kernels which come from right-differentiating narrow kernels in Section 3.2. Their graphs are shown in Figure 5.

$$\frac{d}{dx^+}\Lambda_2^{narr}(x) = \begin{cases} \frac{1}{2}(-1-4x+3x^2) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{6}(-11+12x-3x^2) & \text{for} \quad 1 \le x < 2 \\ 0 & \text{for} \quad 2 \le x \end{cases}$$

$$\frac{d}{dx^+}\Lambda_3^{narr}(x) = \begin{cases} \frac{1}{12}(-4-30x+15x^2+12x^3-5x^4) & \text{for} \quad 0 \le x < 1 \\ \frac{1}{24}(-26-30x+75x^2-36x^3+5x^4) & \text{for} \quad 1 \le x < 2 \\ \frac{1}{120}(-274+450x-255x^2+60x^3-5x^4) & \text{for} \quad 2 \le x < 3 \\ 0 & \text{for} \quad 3 \le x \end{cases}$$

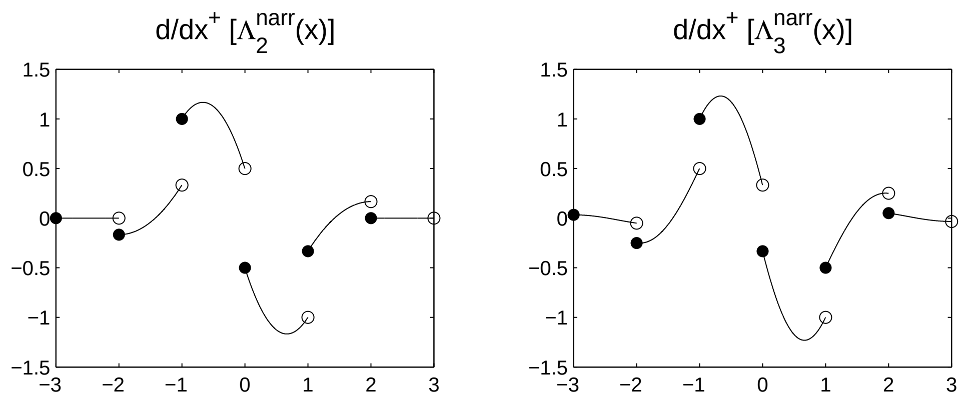

Figure 5: Examples of righ-derivatives of narrow kernels.

7 Numerical examples

7.1 Interpolation in one dimension

Consider the values of $f(x) = \sin(2\pi x)$ given on a uniform grid $y_j = jh$ with $h = 1/n$. We seek to approximate the value of f at the points $x_k = 0.44 + k/(20\sqrt{2})$ for $k = 0$–34. The purpose of this example is to test the performance of various kernels in a particular interpolation problem.

Given n and Λ_h, we compute $(L_0 f)(x_k) = \sum_j f(y_j)\Lambda_h(x_k - y_j)h$ for $k = 0$–34 and we define the error in this approximation as

$$E_n = \max_{0 \le k \le 34} |(L_0 f)(x_k) - f(x_k)|.$$

We use the third-order kernel $\Lambda_2^{sm}(x)$ from Section 3.1 and display the results in Table 4(a). We also show the ratio of consecutive errors so that the order of the approximation can be easily seen. A kernel of order m should result in an error ratio of 2^m. Table 4(b) shows the results using the sixth-order narrow kernel $\Lambda_3^{narr}(x)$ from Section 3.2. Finally, we approximate $f'(x_k)$ using $\frac{d}{dx^+}[\Lambda_3^{narr}(x)]$ and compare it with $f'(x_k) = 2\pi\cos(2\pi x_k)$ using the error defined above. Note that the derivative kernel is discontinuous at the grid nodes but we still expect the derivative to be $O(h^5)$ accurate since $\Lambda_3^{narr}(x)$ is $O(h^6)$. The results are shown in Table 4(c).

	(a)		(b)		(c)	
n	E_n	E_n/E_{2n}	E_n	E_n/E_{2n}	E_n	E_n/E_{2n}
20	6.07456e-04	13.16	4.52503e-06	64.20	2.94629e-04	33.11
40	4.61422e-05	10.40	7.04786e-08	64.03	8.89753e-06	31.28
80	4.43661e-06	8.89	1.10078e-09	61.46	2.84463e-07	30.84
160	4.98824e-07	8.22	1.79106e-11	65.82	9.22460e-09	25.74
320	6.06677e-08		2.72116e-13		3.58444e-10	

Table 4: Results of the one-dimensional interpolation problem. Columns (a) show the error and error ratios for interpolation using the smooth third-order kernel $\Lambda_2^{sm}(x)$. Columns (b) show similar results with the sixth-order narrow kernel $\Lambda_3^{narr}(x)$. Columns (c) show the results for approximating the derivative of the given function using the (discontinuous) first derivative of the narrow kernel from part (b).

7.2 Interpolation in two dimensions

We consider the function $f(x,y) = 4e^{-(x^2+y^2)}\ln(x^2+1)$ in the domain $(x,y) \in [0,2]\times[0,2]$. The function values are computed at the nodes of a uniform square grid of size $h = 2/n$. The goal is to interpolate the function on 100 points equally spaced along the circle of radius 1/6 centered at the point $(1,1)$. The maximum value of $f(x,y)$ is about 1.

In two dimensions we use the product of two interpolation kernels, one in each coordinate direction

$$(L_{0,0}f)(x,y) = \sum_{j=1-R}^{R}\sum_{k=1-R}^{R} f(x_k,y_j)\Lambda_h(x-x_k)\Lambda_h(y-y_j)h^2.$$

As in the previous example, given n and Λ_h, we compute $(L_{0,0}f)(x_k,y_k)$ for $k = 1,\ldots,100$ and we find the error

$$E_n = \max_{0\le k\le 100} |(L_{0,0}f)(x_k,y_k) - f(x_k,y_k)|.$$

For this example, we use two fourth-order kernels. The first one is the smooth kernel $\Lambda_3^{sm}(x)$ from Section 3.1 whose support is $R = 3$ and the second one is the fourth-order narrow kernel $\Lambda_2^{narr}(x)$ from Section 3.2 whose support is $R = 2$. Since both kernels have errors $O(h^4)$, we expect the error ratios E_n/E_{2n} to be about 16 as $h \to 0$. Table 5 shows the results.

7.3 Normal derivative along a curve in 2D

We approximate the normal derivative of the function $u(x,y) = \sin(x)\sin(y)$ at 100 points on the curve

$$C(s) = \left(\frac{1}{2}+\frac{1}{4}\cos(2\pi s), \frac{1}{2}+\frac{1}{4}\sin(4\pi s)\right),$$

(see Figure 6). The points on the curve are $C(i/100)$ for $i = 0,1,\ldots,99$.

	(a)		(b)	
n	E_n	E_n/E_{2n}	E_n	E_n/E_{2n}
10	6.74572e-04	14.07	5.21340e-04	10.64
20	4.79359e-05	16.73	4.89918e-05	17.22
40	2.86496e-06	15.50	2.84454e-06	15.83
80	1.84890e-07	15.75	1.79697e-07	15.87
160	1.17366e-08	16.25	1.13201e-08	14.36
320	7.22040e-10	16.22	7.88351e-10	16.56
640	4.45126e-11	15.39	4.75988e-11	15.59
1280	2.89246e-12		3.05289e-12	

Table 5: Results of the two-dimensional interpolation example. Columns (a) show the error and error ratios for interpolation using the smooth fourth-order kernel $\Lambda_3^{sm}(x)$. Columns (b) show similar results with the fourth-order narrow kernel $\Lambda_2^{narr}(x)$.

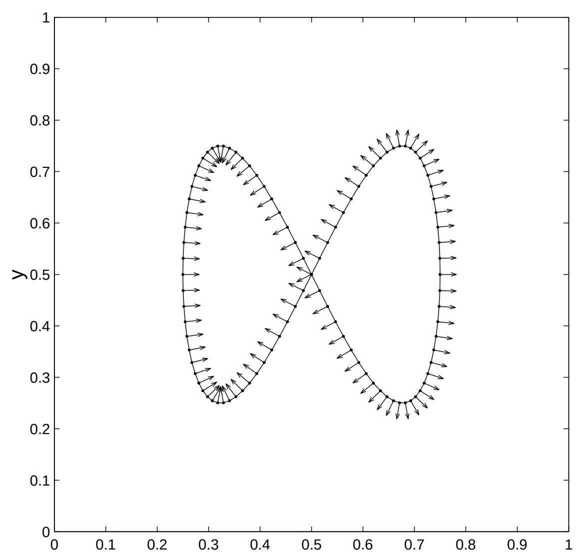

Figure 6: Curve $C(s)$ and its normal vectors at 100 points where the normal derivative of $u(x,y) = \sin(x)\sin(y)$ was approximated from grid values.

In order to use our approximation kernels, the values of $u(x, y)$ are specified on a square grid of dimension $h = 1/n$. Then, an approximation of the partial $u_x(x, y)$ is given by

$$(L_{1,0}u)(x,y) = \frac{1}{h} \sum_{j=1-R_1}^{R_1} \sum_{k=1-R_2}^{R_2} u(x_k, y_j)\Phi_h(x - x_k)\Lambda_h(y - y_j)h^2,$$

where Λ_h, with support R_1, satisfies the moment conditions for interpolation, and Φ_h, with support R_2, satisfies the moment conditions for approximation of the first derivative. The partial u_y is approximated similarly. For this example, the exact normal vectors at the points on the curve were used. Here we seek a fourth-order approximation so we choose $\Lambda_h(x) = \Lambda_3^{sm}(x)$ from Section 3.1 and $\Phi_h(x) = \Lambda_3^{odd}(x)$ from Section 6.1. The error in the approximation was computed as in the previous example. The results are shown in Table 6. Notice that the error ratios are about 2^4, as

n	E_n	E_n/E_{2n}
20	5.17758e-07	15.81
40	3.27539e-08	16.27
80	2.01372e-09	15.93
160	1.26421e-10	10.71
320	1.18054e-11	

Table 6: Errors and error ratios for the computation of the normal derivative of a smooth function using fourth-order kernels.

expected, except when the errors are very small. This is not due to the kernels but to roundoff. In order to evaluate the kernels at a point x, one computes x/h. Due to roundoff error, one expects to get

$$\frac{x+\epsilon}{h+\epsilon} = \frac{x+\epsilon}{h}\left(1 - \frac{\epsilon}{h} + \ldots\right) = \frac{x}{h} + \frac{\epsilon}{h} - x\frac{\epsilon}{h^2} + \ldots$$

so that the largest contribution to roundoff is $O(\epsilon/h^2)$. For a grid of size $h = 1/320$ and $\epsilon = 10^{-16}$ we have that $\epsilon/h^2 \approx 10^{-11}$, which is why the errors saturate around that value.

8 Discussion and conclusion

One of the differences between the two classes of approximation kernels discussed here, smooth and narrow, is their degree of regularity. We have mentioned that for a given order of accuracy and polynomial degree, the narrow kernels have less regularity and smaller support than the smooth kernels (see Table 2). Another difference is that when the interpolation kernels described in Sections 3.1 and 3.2 are evaluated at a grid node, the narrow kernels return the function value at that node. In other words, the narrow kernels satisfy

$$\sum_{k=-\infty}^{\infty} f_k \, \Lambda_h^{narr}(y_i - y_k)h = f_i.$$

On the other hand, the smooth kernels do not have this property. Instead, they return a high-order average of the function values at nearby grid nodes.

The kernels that approximate derivatives of functions also have different properties. The smooth odd kernels of Section 6.1 evaluated at a grid node provide symmetric finite-difference stencils. For example,

$$\Lambda_2^{odd}(y_i) = \frac{1}{12}f_{i-2} - \frac{8}{12}f_{i-1} + \frac{8}{12}f_{i+1} - \frac{1}{12}f_{i+2}.$$

On the other hand, the narrow kernels in Section 6.2 give an asymmetric finite-difference stencil. For example,

$$\frac{d}{dx^+}\Lambda_2^{narr}(y_i) = \frac{1}{6}f_{i-2} - f_{i-1} + \frac{1}{2}f_i + \frac{1}{3}f_{i+1}.$$

We have mentioned already that the latter kernels are discontinuous at the grid nodes. It might be counterintuitive to approximate the derivative of a smooth function with a linear combination of discontinuous kernels. However, these kernels are designed so that for any $0 \leq x < 1$, the Taylor expansions of the approximation $(L_1 f)(x)$ and of $f'(x)$ agree up to the term corresponding to the first non-zero moment of the kernel (excluding $M_0(\Lambda; x)$). In other words, the discontinuity in the kernels appears in the error term of the approximation, and therefore, the discontinuity in the approximation is of the order of the error.

We note that the function $f(x)$ has been assumed to be smooth. In the case that $f(x)$ or one of its derivatives has discontinuities, the arguments presented here no longer apply and the approximations lose accuracy. A specific example of interest is the solution of partial differential equations with singular source terms. In that case, one can regularize the singularity by replacing the delta function in the source term by an interpolation kernel. However, since the weak solution of the PDE is not smooth, the accuracy will be limited unless additional constraints are imposed on the kernels [19, 29]. Such constraints might be one-sided moment conditions [4, 13] or properties that the Fourier transform of the kernel must satisfy [28].

It is common to approximate data with splines. Generally, this technique uses all grid values of the function $f(x)$ to compute a single spline (with appropriate boundary conditions) that spans the data. These splines are global in the sense that a change in one grid value of the function $f(x)$ affects the approximation everywhere. The kernels derived here are based on a different idea even though the smooth kernels in Section 3.1 are themselves splines. The difference is that our kernels are local since they use only a few data points to generate a spline that smoothly approximates a delta function. Therefore, our approximations are unaffected by changes in the grid values of the function outside the support of the kernel.

Several kernels of the type discussed here have been used previously. However, no systematic way of designing the kernels had been proposed. More importantly, we have derived results that clearly state the accuracy that can be achieved by piecewise polynomial kernels for a given smoothness and polynomial degree. Using our results, it is a simple matter to generate the kernels using a symbolic software package.

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References

[1] C. R. Anderson, *A method of local corrections for computing the velocity field due to a distribution of vortex blobs*, J. Comput. Phys. **62** (1986), 111–123.

[2] J. T. Beale and A. Majda, *High order accurate vortex methods with explicit velocity kernels*, J. Comput. Phys. **58** (1985), 188.

[3] D. Benedetto, *Convergence of velicity blobs method for the Euler equation*, Math. Methods. Appl. Sci. **19** (1996), no. 6, 463–479.

[4] R. P. Beyer and R. J. LeVeque, *Analysis of a one-dimensional model for the immersed boundary method*, SIAM J. Numer. Anal. **29** (1992), no. 2, 332–364.

[5] T. F. Buttke, *Velicity methods: Lagrangian numerical methods which preserve the Hamiltonian structure of incompressible fluid flow*, Vortex Flows and Related Numerical Methods (J. T. Beale, G.-H. Cottet, and S. Huberson, eds.), Kluwer Academic Publishers, 1993, NATO ASI Series C, vol. 395, pp. 39–57.

[6] A. Chertock and D. Levy, *Particle methods for dispersive equations*, J. Comput. Phys. **171** (2001), 708–730.

[7] J. P. Choquin and S. Huberson, *Particles simulation of viscous flow*, Comput. Fluids **17** (1989), 397.

[8] R. Cortez, *An impulse-based approximation of fluid motion due to boundary forces*, J. Comput. Phys. **123** (1996), no. 2, 341–353.

[9] ______, *Convergence of high-order deterministic particle methods for the convection-diffusion equation*, Comm. Pure Appl. Math. **50** (1997), no. L, 1235–1260.

[10] G. H. Cottet and P. Koumoutsakos, *Vortex methods: Theory and practice*, Cambridge University Press, 2000.

[11] C. de Boor, *A practical guide to splines*, vol. 27, Springer-Verlag, New York, 1978.

[12] P. Degond and S. Mas-Gallic, *The weighted particle method for convection-diffusion equations part 1: The case of an isotropic viscosity*, Math. Comp. **53** (1989), 485–507.

[13] J. D. Eldredge, A. Leonard, and T. Colonius, *A general deterministic treatment of derivatives in particle methods*, J. Comput. Phys. **180** (2002), no. 2, 686–709.

[14] D. Fishelov, *A new vortex scheme for viscous flows*, J. Comput. Phys. **86** (1990), 211.

[15] G. H. Golub and C. Van Loan, *Matrix computations*, Johns Hopkins University Press, Baltimore, 1983.

[16] O. H. Hald, *Convergence of vortex methods*, Vortex methods and vortex motion (K. E. Gustafson and J. A. Sethian, eds.), SIAM, 1991, pp. 33–58.

[17] P. Koumoutsakos, *Multiscale flow simulations using particles*, Annu. Rev. Fluid Mech. **37** (2005), 457–487.

[18] G. Lacombe and S. Mas-Gallic, *Presentation and analysis of a diffusion-velocity method*, Flows and related numerical methods (Toulouse, 1998) (A. Giovannini, G.-H. Cottet, Y. Gagnon, A. Ghoniem, and E. Maiburg, eds.), vol. 7, ESAIM:Proc., Third International Workshop on Vortex Flows and Related Numerical Methods, Toulouse, France, 1999, pp. 225–233.

[19] R. J. LeVeque and Z. Li, *The immersed interface method for elliptic equations with discontinuous coefficients and singular sources*, SIAM J. Numer. Anal. **31** (1992), no. 2, 332–364.

[20] J. J. Monaghan, *Extrapolating b splines for interpolation*, J. Comput. Phys. **60** (1985), no. 2, 253–262.

[21] ———, *An introduction to sph*, Comp. Phys. Commun. **48** (1988), no. 1, 253–262.

[22] H. Nordmark, *Deterministic high order vortex methods for the 2d Navier-Stokes equation with rezoning*, J. Comput. Phys. **129** (1996), 41–56.

[23] C. S. Peskin, *The immersed boundary method*, Acta Numerica **11** (2002), 479–517.

[24] C. S. Peskin and D. M. McQueen, *A general method for the computer simulation of biological systems interacting with fluids*, Biological Fluid Dynamics (C. P. Ellington and T. J. Pedley, eds.), Cambridge, 1995, Symposia of the Society for Experimental Biology, vol. 19, pp. 265–276.

[25] P. A. Raviart, *An analysis of particle methods*, Numerical Methods in Fluid Dynamics, Springer, 1985, Lecture Notes in Math., vol. 1127, pp. 243–324.

[26] J. T. Becerra Sagredo, *Z-splines: Moment conserving cardinal spline interpolation of compact support for arbitrarily spaced data*, Research No. 2003-10 Seminar für Angewandte Mathematik Eidgenössische Technische Hochschule CH-8092 Zürich Switzerland, 2003.

[27] J. M. Stockie, *Analysis and computation of immersed boundaries, with application to pulp fibres*, Ph.D. thesis, Institute of Applied Mathematics, University of British Columbia, Vancouver, BC, Canada, 1997, http://www.iam.ubc.ca/theses/stockie/stockie.html.

[28] A.-K. Tornberg and B. Engquist, *Regularization techniques for numerical approximation of pdes with singularities*, J. Comput. Phys. **19** (2003), 527–552.

[29] ———, *Numerical approximations of singular source terms in differential equations*, J. Comput. Phys. **200** (2004), 462–488.