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Developing understanding of mathematical modeling in secondary teacher preparation

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Abstract This study examines the evolution of 11 prospective teachers' understanding of mathematical modeling through the implementation of a modeling module within a curriculum course in a secondary teacher preparation program. While the prospective teachers had not previously taken a course on mathematical modeling, they will be expected to include modeling as part of the school curriculum under current state standards. The module consisted of readings, analysis of the Common Core State Standards, carefully designed modeling activities, individual and group work, discussion, presentations, and reflections. The results show that while most prospective teachers had misconceived definitions of mathematical modeling prior to the module, they developed the correct understanding of modeling as an iterative process involving making assumptions and validating conclusions connected to everyday situations. The study reveals how the prospective teachers translated the modeling cycle into practice in the context of a carefully designed open-ended problem and the strong connections between modeling activities and promoting mathematical practices.

Keywords Mathematical modeling · Secondary mathematics · Secondary mathematics pre-service teacher education

Introduction

Historically, mathematical modeling has meant the application of mathematics to solve problems arising in everyday life (Schichl 2004). Mathematical modeling has become an important component of the high school curriculum, where students experience the use of

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mathematics to interpret physical, social, or scientific phenomena. The National Council of Teachers of Mathematics (NCTM) underscores the use of representations to interpret physical, social, and mathematical phenomena in mathematical modeling (NCTM 2000). More explicitly, it states that mathematical models can be used to clarify and interpret the phenomenon and to solve problems. The Common Core State Standards (CCSSM) include "Model with mathematics" as a mathematical practice in K-12 and "mathematical modeling" as a conceptual category in high school (Common Core State Standards Initiative (CCSSI) 2010), and as a result, teacher education in general has developed an increased interest in teacher preparation in mathematical modeling for teaching.

Mathematical modeling problems present unique challenges for teachers, who are not typically required to take courses on modeling as part of their preparation. The challenges stem from working with open-ended problems, making and validating assumptions, and interpreting the mathematical results in the context of the situation given (Blum and Ferri 2009; Blum and Niss 1991). Teaching mathematical modeling requires a full understanding of the practice of modeling as a process in which new perspectives about solving mathematical problems must occur for the students.

In this article, we report findings from a study with secondary prospective teachers on the evolution of their perception of mathematical modeling. The goals of the study were to increase our understanding of the prospective teachers' conceptions of mathematical modeling and how they evolve over the course of the study, and how the stages of the modeling process are reflected in the prospective teachers' solutions of a particular modeling problem. The prospective teachers explored a mathematical modeling problem, "the lost cell phone," that lends itself to the modeling cycle, motivates the content and mathematical practices, and whose context is relatable to students. Our study put into effect a course module aimed at prospective teachers who had no previous formal training in mathematical modeling and tracked the evolution in their understanding of the elements of modeling. Based on the premise that prospective teachers will teach mathematical modeling as part of the school curriculum, it follows that a significant increase in the depth of their understanding of mathematical modeling can later have a positive impact on the quality of their teaching. Three specific research questions that guided the project were:

1. How do prospective teachers translate the modeling cycle into practice in the context of a given problem?
2. How does a mathematical modeling activity promote other mathematical practices?
3. How did the prospective teachers' conception of mathematical modeling evolve throughout the implementation of a mathematical modeling module?

The bigger picture that frames this work is that high-quality instruction requires mathematical content knowledge that is acquired in university-level training and can be further cultivated through systematic reflection on classroom experience (Baumert et al. 2010; see also Ball et al. 2001). Undergraduate mathematics courses offer a natural avenue for providing extended experiences that deepen and develop mathematical knowledge for teaching, including knowledge beyond the curriculum (Zazkis and Mamolo 2011).

We focus specifically on how the modeling process connected with three CCSSM mathematical practices: MP1—make sense of problems and persevere in solving them; MP2—reason abstractly and quantitatively; and MP6—attend to precision. We explore the experiences of prospective teachers throughout the activity and their insights into their thinking on solving "the lost cell phone" problem and their considerations for teaching of modeling to their future high school students.

Theoretical perspectives

Purpose for modeling

Mathematical modeling should distinguish models from the process of modeling. One can consider models as products, while modeling as a process that involves the construction of a model as one of its elements. In particular, building a model is not the same as experiencing the modeling process. Anytime that modeling is used to explain an everyday situation or problem, and the goal(s) of the modeling problem should be considered and made explicit; that is, there should be a purpose for creating models. There are various perspectives on modeling that suggest different purposes (Blomhøj 2009). English (2007) writes that "modeling problems are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artifacts or conceptual tools that are needed for some purpose, or to accomplish some goal" (p. 121). Kaiser and Sriraman (2006) provide a survey of international modeling perspectives, which include the goals of "imparting abilities that enable students to understand central aspects of our world in a better way" as well as the subject-related goals of "structuring of learning processes, introduction of new mathematical concepts, and methods including their illustration."

Models can have some kind of predictive value, which implies that a solution should elicit understanding of what occurred and what will probably follow based on the arrived solution. Furthermore, the outcome should generate discussion of possible modifications of parameters to bring better understanding of the everyday situation. Lesh and Harel (2003) propose that model-eliciting activities come from students' life experiences where mathematics thinking is useful and students can produce symbolic descriptions of the everyday situations.

The modeling cycle

It is well accepted that the modeling process is generally iterative due to the need for assumptions and other choices in order to develop a model. The acceptability of these choices is usually established only after the mathematical results of the model are validated. When the results are unacceptable, a revision of the model is made, leading to a new iteration. The depiction of this process varies in the level of detail of the stages of modeling (e.g., Blum and Leiss 2005; Common Core State Standards Initiative (CCSSI) 2010; Gailbraith and Stillman 2006; Meier 2009; Mooney and Swift 1999; Yoon et al. 2010). When attempting to model a real-world situation, one of the first steps is usually to state a simplified (or idealized) version of the reality, which has been referred to as "Model World" (Mooney and Swift 1999, p. 4) or a "Real Model" (Blum and Leiss 2005, p. 1626) or "Real World Model" (Kaiser and Schwartz 2006, p. 197; Kaiser and Stender 2013, p. 279). The process of simplifying a complex reality into a real-world model has been described in more detail with additional stages (Blum and Leiss 2005). This real-world model forms the basis for a mathematical model of the idealized situation. Further elaboration of iterating through the modeling cycle is discussed by Saeki and Matsuzaki (2011).

While most authors emphasize the use of modeling to interpret "real-world" situations in mathematical formats (English et al. 2005; Sole 2013), in practice, modelers do not restrict themselves to addressing "real-world" problems but also use mathematics to

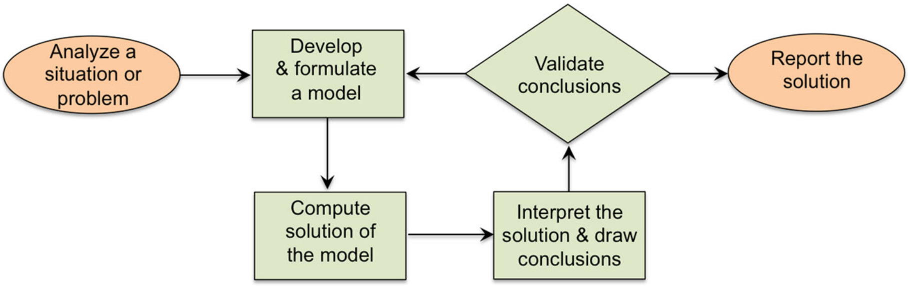

Fig. 1 A general schematic of the modeling cycle process

analyze or predict events from fictional situations (Balicer 2007; Munz et al. 2009; Smith 2014) or purely mathematical scenarios (Common Core State Standards Initiative (CCSSI) 2010, p. 73) such as using fractions to approximate irrational numbers. In these cases, the "real-world" aspect of the modeling process discussed above does not apply and the situation to be modeled may already be idealized while the mathematics side of the process remains intact. For these reasons, we consider a modeling cycle diagram that begins with a situation or problem to be analyzed that may be considered an idealization of reality, a fictional situation, or a purely mathematical situation. The modeling diagram, adapted from the CCSSM, including the validation cycle is shown in Fig. 1.

Several stages of the modeling cycle involve subjective decision-making. Consequently, a particular situation may be modeled in different ways, leading to equally acceptable results. In this research, we explore the choices made by prospective teachers in the context of the lost cell phone problem.

Modeling in the Common Core State Standards in Mathematics

Many authors have offered definitions of what it means to develop a mathematical model and how it fits within the larger process of mathematical modeling (Blum and Niss 1991; Doerr and English 2003; Lesh and Harel 2003; Sole 2013). Since part of our study relates to the connection between the modeling process and the mathematical practices described in the CCSSM, we concentrate on the description of modeling as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (Common Core State Standards Initiative (CCSSI) 2010, p. 72). This definition is consistent with the modeling practice of research mathematicians. The choices mentioned include any assumptions needed to construct a model, the use of informative representations, criteria for validating conclusions, and more. A model may consist of graphs, equations, or functions that describe a phenomenon or seek to explain data on the basis of deeper theoretical ideas (Common Core State Standards Initiative (CCSSI) 2010).

The CCSSM definition of modeling leaves open the opportunity for the modeler to discover and create new mathematics in the process of modeling. In fact, the CCSSM further offer insight into the unpredictable manner in which modeling can be utilized in the curriculum, "Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process" (Common Core State Standards Initiative (CCSSI) 2010, p. 72).

Prospective teacher education in mathematical modeling

According to Baumert et al. (2010), "One of the major findings of qualitative studies on mathematics instruction is that the repertoire of teaching strategies and the pool of alternative mathematical representations and explanations available to teachers in the classroom are largely dependent on the breadth and depth of their conceptual understanding of the subject" (p. 138). Therefore, it is imperative that teachers develop a good understanding of modeling and how to foster students' development of utilizing modeling as they apply their known mathematics and learn new mathematics.

There have been qualitative studies focused on determining the perceptions and performance of prospective teachers on mathematical modeling. Doerr (2007) found that prospective teachers in a mathematical modeling course experienced a shift in their perception of modeling based on their experience in developing mathematical models. Their perspective on modeling evolved from a linear process to a nonlinear cyclic one with multiple possible subcycles. "The findings also suggest that by reflecting on their own modeling activity, pre-service teachers can come to understand the cyclic nature of the modeling process and appreciate the interconnectedness of the cognitive activities involved in the process" (Doerr 2007, p. 73).

Other studies state the need for prospective teachers to experience mathematical modeling in order to develop knowledge about modeling and connected understanding of mathematical content (Cai et al. 2014). An Indonesian case study found that prospective secondary teachers had difficulty stating model assumptions and choices clearly in their work (Widjaja 2013). Studies in Turkey on prospective mathematics teachers before and after completing a university course on modeling indicate that before taking the course, nearly all prospective elementary mathematics teachers could not define mathematical modeling and 24 % expressed that modeling is associated with daily life (Tekin et al. 2011). Eraslan (2011) reports that prospective teachers struggled with the open-endedness of modeling tasks, which were a departure from procedural exercises they were accustomed to in other courses. In other studies, prospective teachers working on modeling activities had intuitive answers but had difficulty formulating a model (Türker et al. 2010) or were able to simplify the given situation and developing a mathematical model but encountered difficulties interpreting the results in context and validating conclusions (Bokova-Güzel 2011).

Research design

The participants

The participants were 11 secondary prospective teachers in a senior-level curriculum and assessment course. The course is part of a secondary teacher preparation program in a mathematics department at a large public university. The prospective teachers are mathematics majors, but none had previously taken a course in mathematical modeling, and one student was concurrently enrolled in a senior-level modeling course.

The research structure and setting

A mathematical modeling module was implemented over six class periods near the end of the semester. The prospective teachers were given an introduction to mathematical modeling and its discussion in the CCSSM, and they explored two mathematical modeling problems with guidelines from the CCSSM and an emphasis on the modeling cycle elements. The mathematical modeling activities were conducted in three teams of three or four prospective teachers whom we refer to hereafter as "PT" followed by a number 1–11. Team 1 consisted of members PT1–PT4, Team 2 consisted of members PT5–PT7, and Team 3 consisted of members PT8–PT11. Their task was to create models for the problems posed, write explanations of their reasoning and their solution, and then present a team poster of their findings to their peers in addition to turning in an individual report. Before an introduction to mathematical modeling as curriculum in the secondary levels, the instructor administered a pre-questionnaire to capture the prospective teachers' conceptions about mathematical modeling and then the exact same questionnaire at the end of the module (we refer to this as the post-questionnaire) to capture their conceptions after engaging in mathematical modeling tasks. Findings were obtained from the responses to the following questions on the questionnaire:

1. One of the standards for mathematical practices is "Model with mathematics." Explain what this means to you.
2. Are modeling with mathematics and solving word problems related? Explain.
3. How can teachers understand and prepare to teach modeling at the middle school and high school levels?
4. What role do you suppose that "real-life" contexts play in modeling problems?

The prospective teachers read the high school-level conceptual category of mathematical modeling and the mathematical practice "Model with mathematics" (MP4) in the CCSSM document, paying special attention to the elements within the modeling cycle explicitly stated within the conceptual category and implicitly stated within the mathematical practice description. Initial class sessions were dedicated to discussions on the demands of the CCSSM in modeling after the prospective teachers read the sections on modeling.

As an introduction to the open-ended nature of mathematical modeling, a first task was posed to the students to engage them in a discussion about necessary background information and assumptions. The problem is from National Council of Teachers of Mathematics (2005):

> A locally owned automated car wash advertises that it serves millions of satisfied customers each year. Is this a reasonable claim?

The prospective teachers worked in small groups to discuss the required information and assumptions to come up with a model, such as the number of minutes it might take for one car to go through the car wash, the number of cars that can get washed in an hour and then in 1 day, the days in a year, the number of hours of operation. Although the problem requires arithmetic aimed at upper elementary levels and proportions aimed at middle school levels, the purpose of posing this problem in class was to experience making decisions that will impact the model, to discuss how to build a model after researching the necessary information, and to discuss reasonable assumptions.

As part of the module in this course, a mathematician whose research is in mathematical modeling was a guest speaker. While presenting a modeling research project, the prospective teachers had the opportunity to ask questions and participate in a discussion with a professional modeler. The instructor of the course served as a mediator during the presentation and discussion by asking questions to the prospective teachers about parallels

Table 1 Lost cell phone modeling problem (Anhalt and Cortez 2015)

A lost cell phone needs to be found. Fortunately, three cell phone towers detect the cell phone signal. Tower 1 detects the signal at a distance of 1072.7 m. Tower 2 detects the signal at a distance of 1213.7 m. Tower 3 detects the signal at a distance of 576.6 m. Based on a coordinate system used by the city, the cell towers are located at (x, y) coordinates as follows: Cell tower 1 is at position (1200, 200) measured in meters from the center of town. Cell tower 2 is at position (800, −450) measured in meters from the center of town. Cell tower 3 is at position (−100, 230) measured in meters from the center of town. Create a model for finding the location of the cell phone. Explain your reasoning

between the research being presented and the process of mathematical modeling as they had been learning.

Following the guest presentation and after exploring several modeling problems and discussing the various stages of mathematical modeling in class, the prospective teacher teams were given the modeling task, "the lost cell phone" (see Table 1). Instructions were given for them to work in independent teams. The instructions for the teams, in addition to creating a model, were to write a description of the team's problem-solving strategies and approaches as they navigated their way through the modeling cycle.

The introductory session on the task required the teams to discuss the problem and think about how they were to approach the problem. They left with a plan on how they were going to proceed with finding background information about cell phone towers and graphing the location of the towers on a coordinate plane.

For the second class session, the prospective teachers came prepared to discuss with all teams how far they had progressed in the modeling process. All the teams arrived to a similar place in the problem, which indicated that they had all gone through the modeling cycle at least once. Detail of the teams' work is described in the findings of this paper. It was during this second class session that each team took a different direction in their modeling process based on reevaluation of their initial assumptions. The teams did not share with each other which direction they were planning to take.

The third class session on this task was dedicated to team presentations of solutions to the lost cell phone problem. The prospective teachers presented the process they went through as they solved the problem, assumptions they made, decisions that followed the assumptions, and their final models. These presentations required team posters and discussions, which allowed the teams to ask questions of each other's solutions, and thus allow comparisons and contrasts of the various approaches. On the same day of the presentations, the prospective teachers submitted individual reports describing their thought process during the teamwork and their reflection of the activity. The reports included assumptions, solutions, reasoning, any revised assumptions, and justification of their results.

Data collection and analysis

This study used qualitative methods in analyzing data from multiple sources, such as team posters, instructor field notes from class discussions, individual written reports with reflections, and pre- and post-questionnaires.

We systematically coded the prospective teachers' reports and reflections by elements in their solutions within the modeling cycle, by mathematical practices in which they engaged, other than 'Model with mathematics' (MP4), and by themes that emerged in the data relevant to the evolution of their conceptualization of mathematical modeling. More specifically, we coded the models produced by the prospective teachers including the solutions to their models and narrative qualitative text that served as explanations to their solutions.

In the analysis of the pre- and post-questionnaires, we used principles of grounded theory method (Strauss and Corbin 1990), allowing us to code the data through the lens of emerging themes. The data were then grouped into similar conceptual themes that helped us understand the participants' thinking and learning of mathematical modeling. We conducted an analysis across data sources and within each source to help us triangulate and understand from various perspectives.

Findings

The findings we report here are categorized by their contribution to the answers of the research questions of the study.

Translating the modeling cycle into practice

The prospective teachers' work on "the lost cell phone" problem reflects the way they experienced the modeling process in this particular problem context. Our findings are grouped according to the first iteration of the modeling cycle, subsequent iterations, and their final reporting of their solution.

Initial cycle in the modeling process

The three teams approached the problem using analytic geometry by plotting three circles with centers at the tower locations and radii given by the distances to the cell phone recorded by the towers. The teams worked toward finding an intersection point of the three circles on a two-dimensional plane, and all prospective teachers reached the same conclusion: The circles do not intersect at a single point (see Fig. 2).

After realizing that there was no single intersection point of all three circles, all teams identified a roughly triangular region whose vertices are the intersection of pairs of circles.

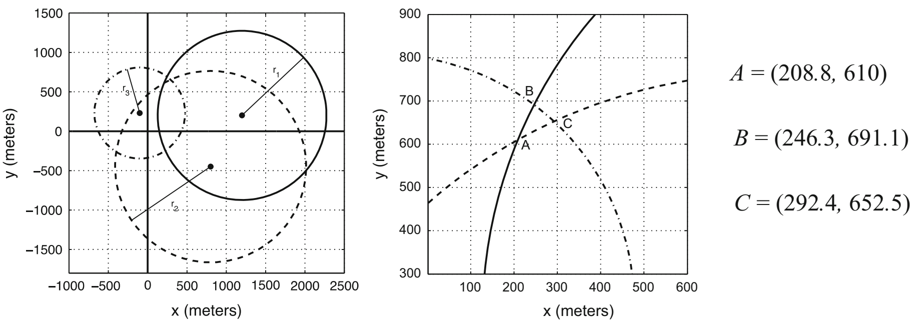

Fig. 2 Region formed by the *three circles* with towers at center of each *circle*—*left* location of the towers and the *circles* of radii equal to the distances to the cell phone, *middle* close-up of the region where the *circles* nearly intersect, and *right* coordinates of the points labeled in the *middle panel*. The distance between points *A* and *C* is about 94 m, and the area of the "*triangular*" region is about 2600 m^2

The area of the triangular region was about 2600 m^2 (see Fig. 2). Several prospective teachers revealed that it was at this precise moment that they realized the difference between a more familiar word problem and a modeling problem (based on the instructor field notes from class discussions and the reflection report by PT7). Initially, the prospective teachers expected this approach to yield the location of the cell phone and be finished. The fact that their first attempt did not produce a single location of the cell phone made the modeling cycle come to life, engaged them at a higher level and motivated them to persevere in finding a better model. All prospective teachers except PT10 indicated in their reflections that their work on the cell phone problem followed closely the description of the CCSSM mathematical practice "Model with mathematics."

Iteration of the mathematical modeling cycle

Assuming that the cell phone and the towers are in a plane led the prospective teachers to a potentially unsatisfying initial solution and encouraged them to reexamine the situation to decide what to do next. It turns out that each team did something different with these findings.

Team 3 decided not to revise the model and report that the cell phone was located somewhere in the 2600 m^2 triangular area shown in Fig. 2. One of the members justified their decision by saying that "…doing any further work or revision of the problem would entail extrapolation of information not given in the problem, and would therefore not serve as a solution to the original problem" (PT8). Another one thought that "the approximate area is 2666 m^2—less than 1 average city block. This may appear to be significant; however in actuality it is not that great and is within the acceptable limits of cell phone triangulation using radio signals" (PT10). Two of the team members considered revisions by writing "I started to think about why the system of equations did not have a unique solution. This mean[t] going back to the assumptions made" (PT9) and "In our model we have a 2666 m^2 triangle that the cell phone could be in. meaning that some of our assumptions might not be correct and we would need more information to rework the model" (PT11). No revised model was produced. There was no indication in the reports of Team 3 members that they iterated through the modeling process, except for PT9 who wrote "Not going through the Basic Modeling Cycle would most likely be tragic for whoever goes through it only once."

Team 2 took the triangular region formed by the three circles centered at the towers and argued that the cell phone would most likely be within a circle through the three vertices of the triangular region. They drew the new circle and gave its center, with coordinates (244, 642), as the likely location of the cell phone (see Fig. 3). Their justification was that "there had to be some amount of error with each tower, and the cell phone had to be somewhere in the region near where the circles appeared to be closest to all intersecting…we determined that the cell phone would most likely be within the circle formed by the three closest points of intersection of the circles" (PT5). One team member wrote: "Therefore, we decided to find a circle that intersected the three points of intersection and its center [point] would be a more accurate representation of the phone's location" (PT6). Their assumption that the distances recorded by the towers had some error allowed for the reported location of the cell phone to be a point near the circles centered at the towers but not necessarily on any one of them. Team 2 members reported having gone through the modeling cycle more than once.

Team 1 also surmised that the lack of a unique intersection point may be attributed to error in the distances to the cell phone measured by the towers. They then made the new

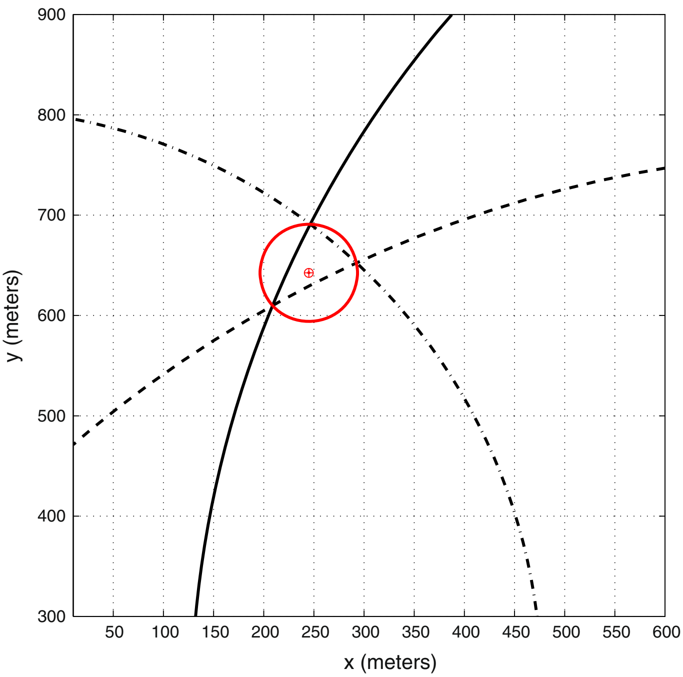

Fig. 3 A *circle* formed by the three intersection points of the three circles

assumption that the errors were up to ±5 % of the distance recorded by each tower and obtained a region where the cell phone is located rather than providing a single point. A member of Team 1 indicated, "So we then had to modify our modeling assumptions to include the possibility that a Margin of Error must be present for the three signals, and that the three signal strengths would have the same margin of error where the signal gets weaker as you get farther from each tower" (PT3). The annuli created by the ±5 % margin of error in the radius of each tower intersect in a trapezoidal region (see Fig. 4), so all points in the trapezoidal area satisfy the assumption and are therefore acceptable. Their revised model became a system of six inequalities (two for each tower): "This gave us a ring of possible locations of the cell phone for each cell tower. We plotted each inequality … in order to see where all three rings intersected" (PT1). Team 1 members reported having gone through the modeling cycle more than once.

Reporting the solution

The final stage of the modeling process is to report the results of the work. The three teams did this in the form of a poster and oral presentation that included the process they went through as they solved the problem, the assumptions made, the decisions that followed the assumptions, and their final model. Although they were given guidance for the presentation, the posters had less guidance, which allowed the prospective teachers to choose what content they thought was important to include on the poster.

Team 3 prepared a poster divided into the sections: Problem Description, Assumptions and Simplifications, Methodology, Results, and Discussion and Future Work. All sections

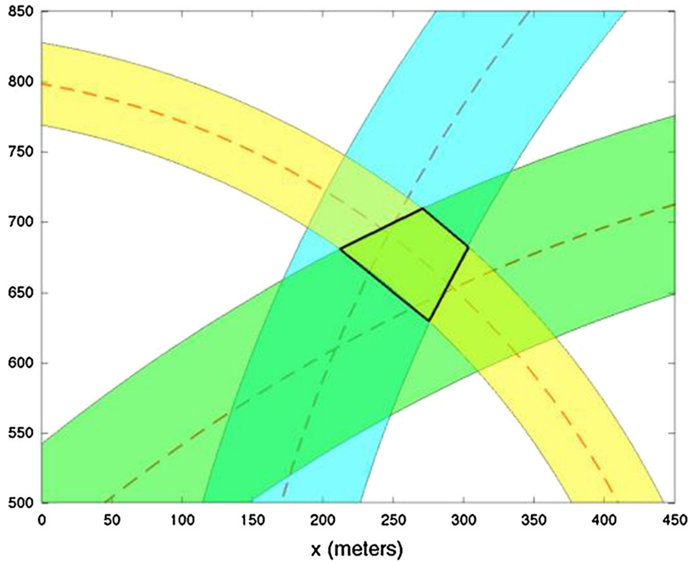

Fig. 4 A region formed by the three bands from the ±5 % margin of error of each of the circles

of this poster included substantial detail, including equations, graphs, and discussion, to follow their process.

Team 2 poster had large graphs and drawings and few words. The only section explicitly included was "Assumptions." The equations of the three circles were displayed early in the poster as was the equation of the circle whose center they reported as the location of the cell phone. One graph showed the three circles centered at the tower locations and a close-up of the region of near intersection with a new circle through the vertices of the triangular region.

Team 1 made a poster with the sections: Assumptions, Decisions, Model, Generalization, and References. Only the first two sections included written descriptions of their choices. The Model section contained the set of inequalities that defined the final trapezoidal region where they concluded the cell phone was located. Two graphs were included in the poster that showed the intersecting annuli centered at the tower locations (one graph was a close-up of the intersecting region). The Generalization section showed the inequalities with tower locations and detection distances written as variables.

The instructor field notes indicate that during the poster presentations the discussion naturally tended to focus on the models themselves, the assumptions made by each team, and the conclusions drawn from the models. The reporting of the solution, as a stage of the modeling process, was not discussed in the same way as the other stages of modeling. Each team made decisions about what was important to include on the posters. The consequence was a large variability in the quality of the posters and indicates a need to more systematically address with students the choices that must be made when reporting results.

Mathematical modeling promotes other mathematical practices

Well-designed mathematical modeling activities have the natural potential to integrate multiple mathematical practices since the modeling process requires students to justify assumptions, validate conclusions, make appropriate mathematical choices, and iterate by refining those choices and assumptions until an acceptable solution is reached. Modeling activities thus provide a setting for connecting content to the CCSSM mathematical practices. Besides "Model with mathematics," we found three other mathematical practices that were prominent across the prospective teachers' work. These were MP1: make sense of problems and persevere in solving them, MP2: reason abstractly and quantitatively, and MP6: attend to precision.

MP1: Make sense of problems and persevere in solving them

The initial elements of the modeling cycle involve analyzing the given situation, identifying essential variables, and making appropriate assumptions to construct a model. All of these and the iterative nature of the process connect directly with sense-making and persevering in solving a problem. By design, the lost cell phone problem statement does not contain the full information to find the exact location of the cell phone. Instead, reasonable solutions can be found under additional assumptions that need to be conceived, tested, and assessed.

Some assumptions that the teams made were that (a) all towers are the same height (but no heights were chosen); (b) the distances from the towers to the cell phones are flat and horizontal (two dimensions); and (c) the level of error in the distances recorded was consistent across the cell phone towers. A first attempt at finding a solution that all prospective teachers made was to think of the tower locations as points on the xy-plane and draw circles centered at those locations with radii equal to the distances recorded by the towers. All prospective teachers demonstrated in their reports some degree of sense-making, especially early in the process, in order to list their assumptions and set up their initial model. They all described that the intersection of the three circles would give the location of the cell phone. They translated this verbal description into equations or graphs to determine the possible location of the cell phone.

One can argue that Teams 1 and 2, who created a revised model to improve their first attempt, showed perseverance in solving the problem. An indication from Team 1 is "my individual thoughts were involving questions like: Are we missing any assumptions? Did we cover all of the necessary variables even though we were not including every possible scenario?" (PT3). The three members of Team 2 indicated perseverance in their reflections. For example, one of them wrote "I found that the multiple approaches to finding the equation for the circle representing the phone's location were highly relevant, as both attempts to find the equation yielded multiple incorrect results while initially calculating them" (PT5). Although Team 3 did not produced a revised model, there were indications that one member used various methods to verify the solution of their model: "To ensure that this graph was correct, we substituted the coordinates of all the points of intersection into the pair of equations they were supposed to satisfy, and verified the approximate equality necessary, as determined by our model" (PT8).

Having studied the mathematical modeling cycle in this course, in general, the prospective teachers recognized the need for understanding the problem deeply and persevering in the creation of a model: "[T]this problem allows students to hypothesize and predict outcomes as well as determine, in the end, how precise of an answer is needed to 'solve' the problem. It opens up a door to ask 'is the problem ever finished?'" (PT6) And, "The fact that the circles don't intersect exactly in one place would make students have to think whether or not their model is accurate enough" (PT7).

MP2: Reason abstractly and quantitatively

Mathematical modeling calls for justifying assumptions and choices made throughout the process, translating between the situation context and the mathematical model, developing a validation rationale, deciding to revise assumptions, and reporting solutions. All of these elements have reasoning at their core. All teams abstracted the situation and represented it as the intersection of three circles. Beyond that, only Team 3 computed the area of the triangular region shown in Fig. 2. Team 1 showed more advanced abstract reasoning by providing a generalization of their final model given arbitrary tower locations and detection distances. After describing that "Our final, generalized model is below, for cell tower 1 at (h_1, k_1) with distance r_1, cell tower 2 at (h_2, k_2) with distance r_2, and cell tower 3 at (h_3, k_3) with distance r_3" (PT2), they provided the set of inequalities describing the region where the cell phone might be in terms of these variables.

MP6: Attend to precision

This practice refers to both the ability to communicate ideas precisely and the accuracy of the numerical answer as necessitated by the problem (Common Core State Standards Initiative (CCSSI) 2010). Both of these are prominent in modeling, as every stage of the modeling cycle requires justification and clarity and the precision in the solution depends on the context. In the lost cell phone problem, after finding that the three circles around the towers did not intersect at a single point, the communication of teams' decisions and the justification of the accuracy of their solutions came to the forefront.

Precision in communication The precision that prospective teachers used in communicating their modeling work was analyzed based on the team posters and the individual written reports. Their writing was classified as being precise or having minor or major imprecise statements. A classification of being precise means that no imprecisions were detected. A classification of having minor or major imprecise statements does not mean that the entire report was imprecise; it means that there was a statement that could be classified as imprecise in a report. Table 2 summarizes the findings and shows that 6 of the 11 prospective teachers used precise communication in their reports, using appropriate units and making statements that show proficiency at interpreting the mathematics in the context of the problem.

A typical example of clear precise language in a report was: "I noticed that the cell phone must be in the set of points that are equidistant from a tower, which is similar to the definition of a circle. If we let the location of each tower be the center of a circle, and use the respective distance as the radius of a circle, we will be graphing three circles which represent the possible location of the cell phone in respect to each tower. Where the circles all intersect is where the distances from the towers agree, and thus is the location of the cell phone" (PT7). This passage includes mathematical terms and connects the empirical

Table 2 Classification of precision in communication in prospective teacher reports

	Precision	Minor imprecision	Major imprecision
Team 1	PT3	PT1, PT4	PT2
Team 2	PT5, PT7	PT6	
Team 3	PT8, PT10, PT11		PT9

situation to the definition of a circle and the cell phone location to the intersection of the three circles. These statements show proficiency at interpreting the mathematics in the context of the problem and vice versa.

Three prospective teachers' communication was deemed to include minor imprecisions. An example by a member of Team 1 is the statement "each equation must be less than if the radius were 1.05 of the detected distance and greater than if the radius were .95 of the detected distance" (PT1). Another example is "After graphing all three circles, we found that there was a spot where all three circles intersected. We assumed that would be the most likely spot where the cell phone was since it would be a spot where all three of the towers would have picked up its signal" (PT6). The use of the word "spot" is ambiguous since it is not clear if it refers to a point or a region. Calling the region "the spot where all three circles intersected" conveys their idea, but is not a mathematically precise statement.

Examples of major imprecisions included referring to a set of inequalities as "equations" (PT2) and indicating that a set of equations "results in a non-homogeneous system, so there is no exact solution" (PT9). Presumably, this prospective teacher intended to say that the system of equations was inconsistent rather than non-homogeneous.

Moschkovich (2012) advocates that the term "precision" in the CCSSM is open to multiple interpretations and that "all students are likely to need time and support for moving from expressing their reasoning and arguments in imperfect form" (p. 22). Although this refers to K-12 students, we argue that this can apply to undergraduate prospective teachers learning a new topic as well or developing understanding in unfamiliar areas of mathematics.

Precision in solutions The standard for mathematical practice "Attend to precision" calls for students to "calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context" (Common Core State Standards Initiative (CCSSI) 2010, p. 7). In this problem, prospective teachers had to decide on the appropriate precision for the cell phone location to report. All prospective teachers discussed the precision they thought was adequate in the context of their models. Members of Team 1 argued that the size of the trapezoidal region they found (see Fig. 4) was small enough for a search to take place. Members of Team 2 were not satisfied with the triangular region resulting from their first model (see Fig. 2) and "decided to improve the model to get a more precise answer" (PT6). Another member of Team 2 commented, "The fact that the circles don't intersect exactly in one place would make students have to think whether or not their model is accurate enough" (PT7), which indicates that the team discussed the precision required in the problem. Team 3 did not produce a revised model and reported that the cell phone would be in the 2600 m^2 triangular region. One team member wrote that this is acceptable since it is "less than one average city block" (PT10). They did, however, discuss modifications to the model: "After accepting the fact that my model did not provide an answer, I looked back to my assumptions and formulation of the problem to see what slight differences in my assumptions could do to my model" (PT8). Two team members discussed possible ways to improve their model such as considering the heights of the towers (PT8 and PT9) and the variations in the landscape (PT9). These ideas were not pursued by the team.

One prospective teacher wrote "this problem requires a student to think about their end result and decide whether it is a decent enough answer or whether more can be done to get a more accurate answer." "This problem allows students to hypothesize and predict outcomes as well as determine, in the end, how precise of an answer is needed to 'solve' the problem. It opens up a door to ask 'is the problem ever finished?'" (PT6). These comments alluded to both precision and perseverance, and how perseverance leads to a better model.

Evolution of prospective teachers' conception of mathematical modeling

The prospective teachers were engaged with mathematical modeling through readings and discussions, and they experienced the modeling process through several problems. This combination of activities resulted in an evolution of the way most of the prospective teachers understood mathematical modeling and in the way they articulated their understanding. These findings are explained below.

General findings

One of the goals of the study was to understand the prospective teachers' conception of mathematical modeling as it evolved in time during the study. We summarize our findings here, which are largely based on the pre- and post-questionnaires and the prospective teachers' reflections on their experience with the "lost cell phone" problem. Table 3 highlights themes that were detected in the prospective teachers' responses to the questionnaire.

Table 3 Questionnaire items and themes that emerged in prospective teacher responses

Themes	PT in pre-Q	PT in post-Q
1. One of the standards for mathematical practices is "Model with mathematics." Explain what this means to you		
Reasonably accurate understanding of mathematical modeling	1, 4, 9	All
Interpreted "Model with mathematics" as synonymous with representation (visual models, manipulative model) or as teacher demonstration	2, 3, 6, 7, 10, 11	None
Showed evolution from pre-Q to post-Q in understanding of Model with mathematics	2, 3, 5, 6, 7, 8, 10, 11	
Mentioned real-world or real-life application	1, 4, 5, 7, 8, 9, 11	All
Mentioned assumptions	None	6
Mentioned or implied integration of multiple concepts	None	3, 7, 10
2. Are modeling with mathematics and solving word problems related? Explain		
Answered "yes"	All	All
Provided a distinction between modeling and word problems	4	4, 5, 7, 8, 9
3. How can teachers understand and prepare to teach modeling at the middle school and high school levels?		
Doing word problems	2, 8	None
Using real-world applications	5, 6, 11	None
Having mathematical modeling experience/knowledge	1, 4, 9	All
Attending teacher workshops or collaborating with other teachers	None	6, 7
4. What role do you suppose that "real-life" contexts play in modeling problems?		
Modeling only applies to real-life situations	2, 7, 11	5, 7, 8
Real-life contexts provide motivation to do math	2, 3, 6, 8	1, 3, 4, 10, 11
Real-life contexts give meaning or relevance to the mathematics	1, 3, 5, 6, 7, 8, 9	5, 9
No clear role specified	4, 10	2, 6

Three major findings from the questionnaire responses were: (1) Without any previous exposure to mathematical modeling, half of the prospective teachers initially misunderstood it as a teacher demonstration model, a visual model, or manipulatives as models. However, at the end of the study, none of the prospective teachers confused mathematical modeling with other interpretations of the word “model;” (2) six prospective teachers showed an improved ability to articulate clearly that modeling is more than problem-solving, involving assumptions and validation; and (3) four prospective teachers went from not associating real-life contexts with mathematical modeling to understanding that the motivation for modeling often comes from real-life settings. A further discussion on each of the findings follows.

Misunderstanding the word “model”

As given in Table 3, the pre-questionnaire shows that six of the prospective teachers incorrectly interpreted “Model with mathematics” as using manipulatives or visual models or thought of it as teacher demonstration. Two prospective teachers did not indicate this misconception but did not have an accurate understanding of modeling, while three prospective teachers already had a reasonably accurate understanding of modeling at the time of the pre-questionnaire. All eight prospective teachers that had an inaccurate understanding of “Model with mathematics” in the pre-questionnaire showed an evolution in their understanding. Of the six that initially confused “model” with a different connotation of the word, one (PT6) mentioned the role of assumptions in the post-questionnaire and three of them (PT3, PT7, and PT10) mentioned that modeling included the integration of multiple mathematics concepts.

The three prospective teachers that showed an accurate understanding of modeling in the pre-questionnaire described modeling similarly in the post-questionnaire and so were classified as not showing evolution in their (already accurate) understanding. As an example, when asked to explain what “Model with mathematics” means to them, one prospective teacher wrote on the pre-questionnaire, “Modeling with mathematics to me is showing students direct ways to think through a problem, whether it be through step-by-step instructions to help them think about their thinking and previous content or through an activity” (PT6). Importantly, by the end of the study none of the prospective teachers confused mathematical modeling with any other interpretation of the words “model” or “modeling.” PT6 wrote in the post-questionnaire that to Model with mathematics means “To be able to use math in contextualized problems. To be able to make assumptions and think reasonably and abstractly about how you can approach and solve a problem.” PT7 wrote in the post-questionnaire that, “Modeling the cell phone problem really helped clarify the modeling process for me, especially because we did it in a group. It was a valuable experience, and now I understand much better what types of problems are considered modeling at the high school level.”

Articulation of the meaning of mathematical modeling

When asked how modeling and word problems are related (question 2), only one prospective teacher (PT4) offered a distinction between modeling and solving word problems in the pre-questionnaire, indicating that word problems “are too structured” relative to modeling. In the post-questionnaire, five prospective teachers offered distinctions: two (PT4 and PT9) mentioned that modeling requires making assumptions while word problems do not, two (PT7 and PT8) mentioned that word problems have a single correct answer, while in modeling there can be multiple acceptable solutions, and one (PT5) explained “that solving a word problem is generally a process in using a model that has already been created.”

Although three of the prospective teachers (PT1, PT4, PT9) had an accurate concept of what mathematical modeling is, the way they articulated it improved over the course of the study. For example, one prospective teacher responded to the question about the relationship between mathematical modeling problems and solving word problems, “… a word problem provides a situation that would be encountered somewhere and sometime and students have to use math to provide a solution by assuming that the problem can be related to a set of math concepts,” and on the post-questionnaire responded, “They can have the same context, but word problems usually give you all the assumptions needed to solve the problem. Modeling is open-ended” (PT9). The six prospective teachers that showed a misunderstanding of mathematical modeling on the pre-questionnaire also reflected the misunderstanding in their responses to the question about how teachers can prepare to mathematical modeling, not surprising since they had a limited understanding of modeling.

In the post-questionnaire, all prospective teachers indicated that teachers can prepare to teach modeling by developing their own experience or knowledge about mathematical modeling, indicating that modeling problems have unique characteristics distinct from traditional word problems. Two prospective teachers mentioned interacting with other teachers about teaching mathematical modeling either as collaboration or in workshops.

Association of “real-life” contexts with mathematical modeling

While mathematical modeling applies more broadly, real-life applications are more common and are often emphasized. In fact, the CCSSM mentions or implies “real life” contexts for modeling about six times in the description of the mathematical practice MP4, “Model with mathematics,” and at least 18 times in the description of the “Modeling” conceptual category for high school (Common Core State Standards Initiative (CCSSI) 2010, pp. 72–73). It is incorrect to equate solving problems from real-world contexts with mathematical modeling since not every problem with real-world context calls for modeling. Also, as mentioned earlier, mathematical modeling can be done within mathematics problems, with fictional contexts, or with real-life situations. Therefore, not all activities from real-world contexts are mathematical modeling problems.

All prospective teachers indicated that “real life” plays a role in mathematical modeling by the end of the study (from question 1 in Table 3). Three prospective teachers indicated that modeling only applies to real-life situations. The notion that modeling only applies to “real-world” situations may be a result of the emphasis of “real-world” or “everyday life” applications in CCSSM sections concerning mathematical modeling.

Two of the six prospective teachers who initially had misconceptions of mathematical modeling (interpreted “Model with mathematics” as synonymous with representation—visual models, manipulative model—or as teacher demonstration) evolved in their thinking from erroneously thinking about modeling to discussing mathematical modeling as pertaining to real life. For example, in the pre-questionnaire, one prospective teacher wrote, “Modeling with math is when it is shown or demonstrated to students by example. So a teacher would model a word problem, by doing the problem while verbalizing the process so students see it in action” and then in the post-questionnaire wrote, “This means to understand math content as it pertains to real life. This is the ability to connect new ideas from math within a context from the real world and applying it” (PT3). Another prospective teacher wrote in the pre-questionnaire that, “Modeling is describing behavior of a system, set, or whatever using mathematical means—equations, graphs, etc." and then in the post-questionnaire wrote, "Students need to be able to use their math skills practically and in real life situations" (PT2).

Discussion

With regard to the lost cell phone problem, the fact that all three teams had the same initial computational findings and reached the correct conclusion that the three circles did not intersect at a common point, set the stage for the teams to decide where to take their models and what assumptions to revise. Interestingly, each of the teams made different decisions at this point. In summary, one team decided that the triangular region was an acceptable answer; another team decided to provide a unique point within this region as a more specific answer. The other team realized that no point in the triangular region actually satisfied their assumptions, so they revised the assumptions (using a margin of error) and found a region of points that did satisfy the new assumptions. The latter two groups' work followed closely the process defined in the modeling cycle of Fig. 1.

The variety in the approaches exemplifies how different students can create different models for the same situation and arrive at acceptable solutions. The solutions differed in the precision with which they located the cell phone. This variability in the precision may have resulted from a lack of discussion early in the modeling process of the accuracy demanded by the problem context.

Iteration through the modeling cycle

The prospective teachers' reflections indicate that Teams 1 and 2 felt that they went through the modeling cycle more than once, and their work supports this. One member of Team 3 (which was satisfied with the triangular region) seemed to indicate that they went through the modeling cycle twice even though this team discussed modified assumptions (such as tower heights) but never developed a modified model. This team seemed uncomfortable introducing new justifiable assumptions because they felt that by doing so, they would change the problem. This may point to a conflict between having to introduce assumptions in modeling problems and using only information provided, as in traditional word problems. It is also possible that this group simply did not want to invest more time in the problem and used their justification to declare their work as finished. We speculate that the inconsistency in believing that they iterated through the modeling cycle may stem from correcting computational errors along the way. Checking and correcting computations may seem like iterating through the modeling cycle when, in fact, is a reflection of the nonlinear nature of the process which is sometimes represented by bidirectional arrows in the cycle diagram (e.g., see Gailbraith and Stillman 2006). Another possibility that may explain why a member of Team 3 had the impression of iterating through the modeling cycle is related to the reevaluation of assumptions in the validation stage. Reflecting on the assumptions initially made, but making no changes, may seem like an iteration. While our study did not settle this issue, it revealed that since the modeling process is not unidirectional, determining the number of cycle iterations in the solution of a problem is not always obvious.

It was evident that Team 1 iterated through the modeling cycle since they modified their assumptions to consider the 5 % margin of error. Although this result yielded a larger area for finding the cell phone than the original area, which may be impractical, the mathematical thinking is accurate because their solution satisfies all of the assumptions of the new model. The evaluation of assumptions was clearly a defining point for this team as they engaged in the iteration process of mathematical modeling.

Team 2, which found the circumcenter of the circle formed by the three points of intersection, did not necessarily modify the original assumptions, but improved their model by making further assumptions. This is an iteration of the modeling cycle since further assumptions were made in addition to the initial assumptions. Although the circumcenter is not a solution to any of the three original equations of circles initially formed from the information given in the problem, this solution seemed intuitive and practical to the team members.

The importance of assumptions in mathematical modeling

One important characteristic of a modeling problem is that its authenticity often results in incomplete information and the need to make informed assumptions based on the interpretation of the information. Each of the prospective teacher teams made a different set of assumptions that led to different models and conclusions. The sharing of results by the teams was very useful for the prospective teachers to experience different acceptable approaches to the same problem. The comparison of the solutions drove the discussion about various assumptions made and strategies used. As each team presented its solution, the impact of their assumptions on the final model became clear. By sharing assumptions during the presentations, the prospective teachers gained insight into each other's thinking of the problem.

Building knowledge of mathematical modeling

With regard to the evolution of the prospective teachers' conception of mathematical modeling, an initial misinterpretation of the words "model" and "modeling" in the context of "mathematical modeling" was not surprising. It is unfortunate that the words "model" and "modeling" are used with different meaning in a variety of settings in mathematics education. Our study showed that without previous exposure, half of the prospective teachers had misconceptions about the meaning of mathematical modeling. We found that a relatively short modeling module within a course can give the prospective teachers the time to develop an understanding of modeling and experience the process.

Mathematical practices motivated by the mathematical modeling process

The CCSSM highlight the need to connect the mathematical practices to mathematical content in mathematics instruction. "…those content standards which set an expectation of understanding are potential 'points of intersection' between the standards for mathematical content and the standards for mathematical practice" (Common Core State Standards Initiative (CCSSI) 2010, p. 8). Mathematical modeling is a unique high school conceptual category that includes content standards only as they relate to other standards, which makes it ideal as a point of intersection. We extend this notion to teacher preparation programs.

In our study, the prospective teachers' work on the modeling process showed the natural use of three mathematical practices: make sense of problems and persevere in solving them, reason abstractly and quantitatively, and attend to precision. The need to engage in these practices arises as part of the modeling cycle, and the prospective teachers' connection to the practices took center stage in their oral presentations, class discussions, and individual reports. Other mathematical practices such as “Look for and make use of structure” (MP7) and “Look for and express regularity in repeated reasoning” (MP8) are likely to come up in the computation of the solution of a model, depending on the specific problem.

Implications and conclusion

Our study followed 11 prospective teachers with no previous experience with mathematical modeling through a compact and focused module on the modeling process that included readings, modeling activities, individual and group work, discussion, and reflection. The combination of activities eventually dispelled misconceptions the prospective teachers had about what mathematical modeling was and provided valuable experiential knowledge that became evident in the evolution of their conceptions of modeling. Such misconceptions are likely to be common in any teacher preparation programs, and therefore, explicit discussions and experiences of mathematical modeling as an iterative process are critical for developing necessary knowledge to effectively implement the mathematical modeling practice in K-12 classrooms. Our module, infused into a mathematics pedagogy course, was intended to go beyond awareness of what it means to do modeling; it successfully broadened and deepened the prospective teachers’ conceptual understanding of mathematical modeling. This is consistent with findings by Doerr (2007) and Cai et al. (2014).

The “lost cell phone” problem was carefully chosen to lead the prospective teachers naturally through the modeling cycle while allowing multiple approaches, different sets of assumptions, and ultimately different acceptable models with different degrees of precision. The need for assumptions in mathematical modeling has implications in teachers’ beliefs about problem-solving in general and the importance of considering the need for relevant assumptions in any problem. This stands in contrast to traditional word problems where all the necessary information is provided, and therefore, when solving these problems, one can proceed from the given information to the goals (a unique solution) of the problem (Zawojewski 2010).

The prospective teacher teams provided solutions and described their work with different degrees of precision. In the classroom, a teacher may find it valuable to discuss the accuracy required for a given modeling activity early in the process so that there is a target precision in the desired solution and assumptions can be appropriately made. In terms of precision in communication, Moschkovich (2012) asserts that regular participation in mathematical discourse provides the time needed for students to develop proficiency in communicating mathematical ideas with more precision.

Although the prospective teachers made progress toward understanding the mathematical modeling process, the connection between mathematical modeling and “real-life” contexts may require more time. This is because a “real-life” context is neither necessary nor sufficient to engage in mathematical modeling. Many word problems address a “real-life” aspect but are stated in ways that do not promote making assumptions or experiencing the modeling cycle (Tam 2011). This nuance did not come across clearly in the prospective teachers’ work.

Building teacher knowledge of the modeling process requires careful integration of mathematical modeling into teacher preparation coursework. This may need to consider the prospective teachers’ background knowledge and possible misconceptions regarding modeling. It is worthwhile for teacher preparation programs to reexamine their curricula and consider mathematical modeling as a focus for developing content knowledge and mathematical practices as recommended by The Mathematical Education for Teachers II Report (Conference Board of the Mathematical Sciences 2012). It was important that the prospective teachers engage in this module, as it was their only opportunity during their teacher preparation to be exposed to mathematical modeling as they are expected to teach. The experience offered by mathematical modeling modules of this type can have a lasting positive effect in the prospective teachers' quality of instruction. There is a need to engage future teachers and practicing teachers in mathematical modeling so that K-12 students experience mathematical modeling as part of their mathematics education. More targeted research is necessary to address the development of future teachers' knowledge of mathematical modeling for teaching, especially in areas of opportunities and improvement as found in this research study. Teacher preparation work in courses pertaining to mathematics curriculum is a multilayer endeavor. The modeling problems explored should provide prospective teachers with the experiences of doing mathematics as learners of mathematics and then extend them further to a professional level to examine the mathematics in the context of the curriculum and student expectations.

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